HOCHSCHILD COHOMOLOGY OF PROJECTIVE HYPERSURFACES

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ABSTRACT. We compute Hochschild cohomology of projective hypersurfaces starting from the Gerstenhaber-Schack complex of the (restricted) structure sheaf. We are particularly interested in the second cohomology group and its relation with deformations. We show that a projective hypersurface is smooth if and only if the classical HKR decomposition holds for this group. In general, the first Hodge component describing scheme deformations has an interesting inner structure corresponding to the various ways in which first order deformations can be realized: deforming local multiplications, deforming restriction maps, or deforming both. We make our computations precise in the case of quartic hypersurfaces, and compute explicit dimensions in many examples.

1. Introduction

Hochschild cohomology originated as a cohomology theory for associative algebras, which is known to be closely related to deformation theory since the work of Gerstenhaber. Meanwhile, both the cohomology and the deformation side of the picture have been developed for a variety of mathematical objects, ranging from schemes [22] [15] to abelian [18], [17] and differential graded [14], [16] categories. One of the first generalizations considered after the algebra case was the case of presheaves of algebras, as thoroughly investigated by Gerstenhaber and Schack [7], [9], [10]. For a presheaf $\mathcal{A}$, Hochschild cohomology is defined as an Ext of bimodules $\mathrm{Ext}_{\mathcal{A}\text{-}\mathcal{A}}(\mathcal{A},\mathcal{A})$ in analogy with the algebra case. An important tool in the study of this cohomology is the (normalized, reduced) Gerstenhaber-Schack double complex $\mathbf{C}(\mathcal{A})$. We denote its associated total complex by $\mathbf{C}_{\mathrm{GS}}(\mathcal{A})$, and the cohomology of this complex by $H^n_{\mathrm{GS}}(\mathcal{A}) = H^n\mathbf{C}_{\mathrm{GS}}(\mathcal{A})$. We have $H^n_{\mathrm{GS}}(\mathcal{A}) \cong \mathrm{Ext}^n_{\mathcal{A}\text{-}\mathcal{A}}(\mathcal{A},\mathcal{A})$. Unlike what the parallel result for associative algebras may lead one to expect, in general $H^2_{\mathrm{GS}}(\mathcal{A})$ is not identified with the family of first order deformations of the presheaf $\mathcal{A}$. A correct interpretation of $H^2_{\mathrm{GS}}(\mathcal{A})$ is as the family of first order deformations of $\mathcal{A}$ as a *twisted* presheaf, and an explicit isomorphism

$$H^2_{\mathrm{GS}}(\mathcal{A}) \longrightarrow \mathrm{Def}_{\mathrm{tw}}(\mathcal{A}) \tag{1.1}$$

is given in [6, Thm. 2.21]. Moreover, in loc. cit., if $\mathcal{A}$ is quasi-compact semi-separated, the existence of a bijective correspondence between the first order deformations of $\mathcal{A}$ as a twisted presheaf and the abelian deformations of the category $\mathsf{Qch}(\mathcal{A})$ of quasi-coherent sheaves is proven. Hence in this case there are isomorphisms $H^2_{\mathrm{GS}}(\mathcal{A}) \cong \mathrm{Def}_{\mathrm{tw}}(\mathcal{A}) \cong \mathrm{Def}_{\mathrm{ab}}(\mathsf{Qch}(\mathcal{A}))$.

Throughout, let k be an algebraically closed field of characteristic zero. Of particular interest is the case where $\mathcal{A}$ is a presheaf of commutative k-algebras over a poset or more generally a small category. As discussed in [7], in this case the complex $\mathbf{C}_{\mathrm{GS}}(\mathcal{A})$ admits the Hodge decomposition of complexes

$$\mathbf{C}_{\mathrm{GS}}(\mathcal{A}) = \bigoplus_{r\in\mathbb{N}} \mathbf{C}_{\mathrm{GS}}(\mathcal{A})_r, \tag{1.2}$$

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which induces the Hodge decomposition of the cohomology groups $H^n_{\mathrm{GS}}(\mathcal{A})$ in terms of the cohomology groups $H^n_{\mathrm{GS}}(\mathcal{A})_r = H^n\mathbf{C}_{\mathrm{GS}}(\mathcal{A})_r$:

$$H^n_{\mathrm{GS}}(\mathcal{A}) = \bigoplus_{r\in\mathbb{N}} H^n_{\mathrm{GS}}(\mathcal{A})_r. \tag{1.3}$$

The zero-th Hodge complex $\mathbf{C}_{\mathrm{GS}}(\mathcal{A})_0$ is nothing but the simplicial cohomology complex of $\mathcal{A}$, and the first Hodge complex $\mathbf{C}_{\mathrm{GS}}(\mathcal{A})_1$, which is called the asimplicial Harrison complex in [7], classifies first order deformations of $\mathcal{A}$ as a commutative presheaf. Hence, in this case the map (1.1) naturally restricts to

$$H^2_{\mathrm{GS}}(\mathcal{A})_1 \longrightarrow \mathrm{Def}_{\mathrm{cpre}}(\mathcal{A}). \tag{1.4}$$

Let $(X, \mathcal{O}_X)$ be a quasi-compact separated scheme with an affine open covering $\mathfrak{V}$ which is closed under intersection, and let $\mathcal{A} = \mathcal{O}_X|_{\mathfrak{V}}$ be the restriction of $\mathcal{O}_X$ to the covering $\mathfrak{V}$. The cohomology $H^\bullet_{\mathrm{GS}}(\mathcal{A})$ turns out to be isomorphic to the Hochschild cohomology

$$HH^\bullet(X) := \mathrm{Ext}^\bullet_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X)$$

of the scheme X where $\Delta\colon X \to X \times X$ is the diagonal map [17]. If furthermore, X is smooth, then the Hodge decomposition corresponds to the HKR decomposition and we obtain the familiar formula

$$HH^n(X) \cong \bigoplus_{p+q=n} H^p(X, \wedge^q \mathcal{T}_X) \tag{1.5}$$

where $\mathcal{T}_X$ is the tangent sheaf of X. This formula has been proved in various different contexts and ways [9], [15], [22], [24], [6].

The decomposition (1.5) has been generalized to the not necessarily smooth case by Buchweitz and Flenner in [4], using the Atyiah-Chern character. In terms of the relative cotangent complex $\mathbb{L}_{X/k}$, the generalization is given by

$$HH^n(X) \cong \bigoplus_{p+q=n} \mathrm{Ext}^p_X(\wedge^q \mathbb{L}_{X/k}, \mathcal{O}_X) \tag{1.6}$$

where $\wedge^q$ should be understood as derived exterior product. Their arguments are mostly established in the derived category $\mathbf{D}(X)$, and an interpretation of cohomology classes in terms of GS-representatives is not immediate.

Since we need GS-representatives in order to use the deformation interpretation from (1.1), our starting point in this paper is the Gerstenhaber-Schack complex $\mathbf{C}_{\mathrm{GS}}(\mathcal{A})$. In case $\mathcal{A} = \mathcal{O}_X|_{\mathfrak{V}}$ for a projective hypersurface X, in §4.1 we construct a smaller complex $\mathcal{H}^\bullet$ and we give an explicit quasi-isomorphism $\mathcal{H}^\bullet \to \mathbf{C}_{\mathrm{GS}}(\mathcal{A})$. Our construction of $\mathcal{H}^\bullet$ builds on [2] and [19], in both of which the Hochschild (co)homology of affine hypersurfaces is computed. Following their methods, in §3 we describe the Hodge components of the affine Hochschild cohomology groups in terms of the cotangent complex. The other key ingredient in our approach to the projective case is the use of a mixed complex associated to a pair of orthogonal sequences in a commutative ring, which is developed in the self-contained section §2.

In §4.2 we present the cotangent complex $\mathbb{L}_{X/k}$ in terms of twisted structure sheaves $\mathcal{O}_X(l)$, and we verify that the cohomology of $\mathcal{H}^\bullet$ agrees with (1.6), and $\mathcal{H}^\bullet$ can be considered to be a natural enhancement of (1.6). It is of interest whether the Hodge and generalized HKR decompositions are component-wise isomorphic for a general variety. Since they agree for smooth varieties and for hypersurfaces, it is reasonable to expect the answer to be positive.

In general however, we have not yet devised an efficient method to obtain such a nice $\mathcal{H}^\bullet$, relating Gerstenhaber-Schack cohomology and Čech cohomology. For this we seem to lack smaller projective resolutions on affine pieces which are easily computable. In the present case, projective hypersurfaces are tractable using [2] and [19] as well as our technical results from §2. Moreover, $\mathcal{H}^\bullet$ is the total Čech complex of a complex of sheaves, making our computation feasible.

In §5, we compute the cohomology groups of $\mathcal{H}^\bullet$ in terms of two easier complexes $\mathcal{C}^\bullet(\boldsymbol{u}; S)$ and $\mathcal{K}^\bullet(\boldsymbol{v}; R)$ of graded modules. Our main theorem is the following:

Theorem 1.1. *Let $X \subset \mathbb{P}^n$ be a projective hypersurface of degree d. Denote by P^i the i-th cohomology group of $\mathcal{C}^\bullet(\boldsymbol{u}; S)$ and by Q^i the i-th cocycle group. Denote by Z^i the i-th cocycle group of $\mathcal{K}^\bullet(\boldsymbol{v}; R)$. Then the cohomology of $\mathcal{H}^\bullet$ is given by*

(1) *when $d > n+1$,*

$$H^i(\mathcal{H}^\bullet) \cong \bigoplus_{r<i} P^{i-2r}_{r+(i-r)(d-1)} \oplus Q^{-i}_i \oplus \mathscr{S}(Z^{-i+n-1}_{d-i-2});$$

(2) *when $d = n+1$,*

$$H^i(\mathcal{H}^\bullet) \cong \begin{cases} \bigoplus_{r<i} P^{i-2r}_{r+n(i-r)} \oplus Q^{-i}_i, & i \neq n-1, n, \\ \bigoplus_{r<i} P^{i-2r}_{r+n(i-r)} \oplus Q^{-i}_i \oplus k^n, & i = n-1, \\ \bigoplus_{r\leq i} P^{i-2r}_{r+n(i-r)}, & i = n; \end{cases}$$

(3) *when $d < n+1$,*

$$H^i(\mathcal{H}^\bullet) \cong \bigoplus_{r<i} P^{i-2r}_{r+(i-r)(d-1)} \oplus Q^{-i}_i.$$

In the above formulas, $\mathscr{S}$ is a linear map defined in (5.2), and the subscripts of $P^\bullet$, $Q^\bullet$, $Z^\bullet$ stand for the degrees of homogeneous elements in $P^\bullet$, $Q^\bullet$, $Z^\bullet$.

In §6, we give some applications of Theorem 1.1. As a first application, we give a cohomological characterization of smoothness for projective hypersurfaces in §6.1. Recall that an affine hypersurface $\mathrm{Spec}(A)$ is smooth if and only if the first Hodge component $H^2_{(1)}(A, A)$ vanishes (Remark 3.2). In deformation theoretic terms, this corresponds to the fact that A has only trivial commutative deformations. For a projective hypersurface X with restricted structure sheaf $\mathcal{A} = \mathcal{O}_X|_{\mathfrak{V}}$, the parallel statement is that X is smooth if and only if the first Hodge component $H^2_{\mathrm{GS}}(\mathcal{A})_1$ coincides with its subgroup $H^1(X, \mathcal{T}_X)$ which describes locally trivial scheme deformations of X. In other words, X is smooth if and only if the classical HKR decomposition (1.5) holds for the second Hochschild cohomology group of X (Theorem 6.3). In the appendix A we present a more general proof of this converse HKR theorem for complete intersections making use of global generation of the normal sheaf, which was suggested to us by the referee.

Next we look into the fine structure of the first Hodge component $H^2_{\mathrm{GS}}(\mathcal{A})_1$. Recall that a GS n-cochain has $n+1$ components coming from the double complex $\mathbf{C}(\mathcal{A})$, in particular

$$\mathbf{C}^2_{\mathrm{GS}}(\mathcal{A}) = \mathbf{C}^{0,2}(\mathcal{A}) \oplus \mathbf{C}^{1,1}(\mathcal{A}) \oplus \mathbf{C}^{2,0}(\mathcal{A}). \tag{1.7}$$

Following [6], we usually write a GS 2-cochain as (m, f, c) corresponding to the decomposition (1.7). In §6.1, we show that for $\mathcal{A} = \mathcal{O}_X|_{\mathfrak{V}}$ with X a projective hypersurface of dimension ≥ 2, there exists a complement E of $H^1_{\mathrm{simp}}(\mathfrak{V}, \mathcal{T})$ inside $H^2_{\mathrm{GS}}(\mathcal{A})_1$ consisting of Hochschild classes of the form $[(m, 0, 0)]$. Intuitively, we visualize the situation with the aid of the following diagram:

$$\begin{array}{lcccc}
\text{Hodge components:} & H^2_{\mathrm{GS}}(\mathcal{A})_2 & & H^2_{\mathrm{GS}}(\mathcal{A})_1 & H^2_{\mathrm{GS}}(\mathcal{A})_0 \\
 & \wr & \swarrow\!\!\wr & \wr & \wr \\
\text{HKR components:} & H^0_{\mathrm{simp}}(\mathfrak{V}, \wedge^2\mathcal{T}) & E & H^1_{\mathrm{simp}}(\mathfrak{V}, \mathcal{T}) & H^2_{\mathrm{simp}}(\mathfrak{V}, \mathcal{A}) \\
 & \downarrow & \swarrow & \downarrow & \downarrow \\
\text{representatives:} & (m, 0, 0) & & (0, f, 0) & (0, 0, c)
\end{array}$$

In general, we call a Hochschild 2-class *intertwined* if it cannot be written as a sum of the form $[(m, 0, 0)] + [(0, f, 0)]$. Intertwined classes are interesting from the point of view of deformation theory, as the only way to realize such a class is by simultanous non-trivial deformation of local multiplications and of restriction maps, with neither deforming only the multiplications, nor deforming only the restriction maps leading to a well-defined deformation. Remarkably, based upon the results from §5, an intertwined 2-class can only exist for a non-smooth projective curve in $\mathbb{P}^2$ of degree ≥ 5, and we give concrete examples of such curves of degree ≥ 6 in §6.2. We leave the existence of intertwined 2-classes for degree 5 curves as an open question.

In §6.3, we study the case when X is a quartic surface in $\mathbb{P}^3$ in some detail. We show that the dimension of $H^2_{\mathrm{GS}}(\mathcal{A})_1$ lies between 20 and 32, reaching all possible values except 30 and 31. The minimal value $H^2_{\mathrm{GS}}(\mathcal{A})_1 = 20$ is reached in the smooth case, in which X is a K3 surface and $H^2_{\mathrm{GS}}(\mathcal{A})_1 \cong H^1(X, \mathcal{T}_X)$, as well as in some non-smooth examples like the Kummer surfaces. Further, we discuss the fine structure of $H^2_{\mathrm{GS}}(\mathcal{A})_1$ in several examples. Finally, let us mention that the zero-th Hodge component $H^2_{\mathrm{GS}}(\mathcal{A})_0$ is invariably one dimensional, and we know that the dimension of the second Hodge component $H^2_{\mathrm{GS}}(\mathcal{A})_2$ is at least one. Although our results allow us to compute the dimension of $H^2_{\mathrm{GS}}(\mathcal{A})_2$ in concrete examples, so far we have not determined the precise range of this dimension.

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2. Mixed complexes associated to orthogonal sequences

This section is self-contained. In order to make preparations for future computations, we construct several complexes which are related to Koszul complexes, as well as quasi-isomorphisms between them.

Let R be a commutative ring, and let $\boldsymbol{u} = (u_0 \dots, u_n)$, $\boldsymbol{v} = (v_0, \dots, v_n)$ be two sequences in R. We call $(\boldsymbol{u}, \boldsymbol{v})$ a *pair of orthogonal sequences of length n (an n-POS)* if

$$\sum_{i=0}^{n} u_i v_i = 0$$

holds in R. Let $(\mathcal{K}^\bullet(\boldsymbol{u}; R), \partial_{\boldsymbol{u}})$ be the Koszul cochain complex determined by $\boldsymbol{u}$, namely, $\mathcal{K}^\bullet(\boldsymbol{u}; R)$ is the DG R-algebra $\wedge^\bullet(Re_0 \oplus \cdots \oplus Re_n)$ with $|e_i| = -1$ and $\partial_{\boldsymbol{u}}(e_i) = u_i$. Similarly, let

$(\mathcal{K}_\bullet(\boldsymbol{v};R),\partial^{\boldsymbol{v}}) = \wedge^\bullet(Rf_0\oplus\cdots\oplus Rf_n)$ be the Koszul chain complex determined by $\boldsymbol{v}$. Applying $\operatorname{Hom}_R(-,R)$ to $\mathcal{K}_\bullet(\boldsymbol{v};R)$, we obtain a cochain complex $\operatorname{Hom}_R^\bullet(\mathcal{K}_\bullet(\boldsymbol{v};R),R)$ whose terms are

$$\operatorname{Hom}_R^{-p}(\mathcal{K}_\bullet(\boldsymbol{v};R),R) = \operatorname{Hom}_R(\mathcal{K}_p(\boldsymbol{v};R),R) = \bigoplus_{0\le i_1<\cdots<i_p\le n} R(f_{i_1}\wedge\cdots\wedge f_{i_p})^*$$

and whose differentials are

$$\begin{aligned}(\partial^{\boldsymbol{v}})^*\colon \operatorname{Hom}_R^{-p}(\mathcal{K}_\bullet(\boldsymbol{v};R),R) &\longrightarrow \operatorname{Hom}_R^{-p-1}(\mathcal{K}_\bullet(\boldsymbol{v};R),R)\\ (f_{i_1}\wedge\cdots\wedge f_{i_p})^* &\longmapsto \sum_{j=0}^n v_j(f_j\wedge f_{i_1}\wedge\cdots\wedge f_{i_p})^*.\end{aligned}$$

For each p, the correspondence $e_{i_1}\wedge\cdots\wedge e_{i_p}\longleftrightarrow(f_{i_1}\wedge\cdots\wedge f_{i_p})^*$ establishes an isomorphism between $\mathcal{K}^{-p}(\boldsymbol{u};R)$ and $\operatorname{Hom}_R^{-p}(\mathcal{K}_\bullet(\boldsymbol{v};R),R)$ in a natural way. The differentials $(\partial^{\boldsymbol{v}})^*$ induce another complex structure on $\mathcal{K}^\bullet(\boldsymbol{u};R)$ given by

$$\begin{aligned}\partial_{\boldsymbol{v}}\colon \mathcal{K}^{-p}(\boldsymbol{u};R) &\longrightarrow \mathcal{K}^{-p-1}(\boldsymbol{u};R)\\ e_{i_1}\wedge\cdots\wedge e_{i_p} &\longmapsto \sum_{j=0}^n v_je_j\wedge e_{i_1}\wedge\cdots\wedge e_{i_p}.\end{aligned}$$

Remark 2.1. $(\mathcal{K}^\bullet(\boldsymbol{u};R),\partial_{\boldsymbol{v}})$ is isomorphic to the Koszul complex determined by the sequence $\boldsymbol{v}^\star=(v_0,-v_1,\ldots,(-1)^nv_n)$.

The following lemma is very easy to prove.

Lemma 2.1. $\mathcal{K}^\bullet(\boldsymbol{u},\boldsymbol{v};R) = (\mathcal{K}^\bullet(\boldsymbol{u};R),\partial_{\boldsymbol{u}},\partial_{\boldsymbol{v}})$ *is a mixed complex.*

This mixed complex gives rise to a double complex $\mathcal{K}^{\bullet,\bullet}(\boldsymbol{u},\boldsymbol{v};R)$ in the first quadrant as in Figure 1. For $r\in\mathbb{N}$, define $\tau^r\mathcal{K}^{\bullet,\bullet}(\boldsymbol{u},\boldsymbol{v};R)$ to be the quotient double complex of $\mathcal{K}^{\bullet,\bullet}(\boldsymbol{u},\boldsymbol{v};R)$ consisting of all entries whose coordinates satisfy $0\le q\le r$.

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \iddots\\
\uparrow{\scriptstyle\partial_{\boldsymbol{v}}} & & \uparrow{\scriptstyle\partial_{\boldsymbol{v}}} & & \uparrow{\scriptstyle\partial_{\boldsymbol{v}}} & & \uparrow{\scriptstyle\partial_{\boldsymbol{v}}} & & \\
\mathcal{K}^{-3}(\boldsymbol{u},\boldsymbol{v};R) & \xrightarrow{\partial_{\boldsymbol{u}}} & \mathcal{K}^{-2}(\boldsymbol{u},\boldsymbol{v};R) & \xrightarrow{\partial_{\boldsymbol{u}}} & \mathcal{K}^{-1}(\boldsymbol{u},\boldsymbol{v};R) & \xrightarrow{\partial_{\boldsymbol{u}}} & \mathcal{K}^{0}(\boldsymbol{u},\boldsymbol{v};R) & & \\
\uparrow{\scriptstyle\partial_{\boldsymbol{v}}} & & \uparrow{\scriptstyle\partial_{\boldsymbol{v}}} & & \uparrow{\scriptstyle\partial_{\boldsymbol{v}}} & & & & \\
\mathcal{K}^{-2}(\boldsymbol{u},\boldsymbol{v};R) & \xrightarrow{\partial_{\boldsymbol{u}}} & \mathcal{K}^{-1}(\boldsymbol{u},\boldsymbol{v};R) & \xrightarrow{\partial_{\boldsymbol{u}}} & \mathcal{K}^{0}(\boldsymbol{u},\boldsymbol{v};R) & & & & \\
\uparrow{\scriptstyle\partial_{\boldsymbol{v}}} & & \uparrow{\scriptstyle\partial_{\boldsymbol{v}}} & & & & & & \\
\mathcal{K}^{-1}(\boldsymbol{u},\boldsymbol{v};R) & \xrightarrow{\partial_{\boldsymbol{u}}} & \mathcal{K}^{0}(\boldsymbol{u},\boldsymbol{v};R) & & & & & & \\
\uparrow{\scriptstyle\partial_{\boldsymbol{v}}} & & & & & & & & \\
\mathcal{K}^{0}(\boldsymbol{u},\boldsymbol{v};R) & & & & & & & &
\end{array}$$

FIGURE 1. Double complex $\mathcal{K}^{\bullet,\bullet}(\boldsymbol{u},\boldsymbol{v};R)$

Suppose that v_t is invertible for some $t\in\{0,1,\ldots,n\}$. Let $\boldsymbol{w}=(u_0,\ldots,\widehat{u_t},\ldots,u_n)$, and $(\mathcal{K}^\bullet(\boldsymbol{w};R),\partial_{\boldsymbol{w}})$ be the corresponding Koszul complex. Define $\iota\colon\mathcal{K}^\bullet(\boldsymbol{w};R)\to\mathcal{K}^\bullet(\boldsymbol{u};R)$ to be the canonical embedding morphism, and define $\pi\colon\mathcal{K}^\bullet(\boldsymbol{u};R)\to\mathcal{K}^\bullet(\boldsymbol{w};R)$ by

$$\pi(e_{i_1}\wedge\cdots\wedge e_{i_p}) = \begin{cases} e_{i_1}\wedge\cdots\wedge e_{i_p}, & \text{if none of } i_j \text{ is } t,\\[2ex] -\displaystyle\sum_{k\neq t} v_kv_t^{-1}e_{i_1}\wedge\cdots\wedge e_{i_{j-1}}\wedge e_k\wedge e_{i_{j+1}}\wedge\cdots\wedge e_{i_p}, & \text{if } t=i_j \text{ for some } j.\end{cases}$$

for each p.

It is routine to prove that $\partial_{\boldsymbol{w}}\pi(e_{i_1}\wedge\cdots\wedge e_{i_p})=\pi\partial_{\boldsymbol{u}}(e_{i_1}\wedge\cdots\wedge e_{i_p})$. Hence we have

Lemma 2.2. *$\pi\colon\mathcal{K}^\bullet(\boldsymbol{u};R)\to\mathcal{K}^\bullet(\boldsymbol{w};R)$ is a morphism of complexes.*

Lemma 2.3. *For all p, the sequence*

$$0\longrightarrow\mathcal{K}^{-p+1}(\boldsymbol{w};R)\xrightarrow{\partial_{\boldsymbol{v}}\iota}\mathcal{K}^{-p}(\boldsymbol{u};R)\xrightarrow{\pi}\mathcal{K}^{-p}(\boldsymbol{w};R)\longrightarrow 0$$

is split exact.

Proof. First of all, this is indeed a complex since $\pi\partial_{\boldsymbol{v}}\iota=0$.

Next, we consider the map $\mathrm{id}-\iota\pi$. By the definition of π, if none of i_j is t, then $(\mathrm{id}-\iota\pi)(e_{i_1}\wedge\cdots\wedge e_{i_p})=0$; if $t=i_j$, then $(\mathrm{id}-\iota\pi)(e_{i_1}\wedge\cdots\wedge e_{i_p})=\partial_{\boldsymbol{v}}((-1)^{j-1}v_t^{-1}e_{i_1}\wedge\cdots\wedge\widehat{e_{i_j}}\wedge\cdots\wedge e_{i_p})$. It follows that there exists a map $\zeta\colon\mathcal{K}^{-p}(\boldsymbol{u};R)\to\mathcal{K}^{-p+1}(\boldsymbol{w};R)$ given by

$$\zeta(e_{i_1}\wedge\cdots\wedge e_{i_p})=\begin{cases}0, & \text{if none of } i_j \text{ is } t,\\ (-1)^{j-1}v_t^{-1}e_{i_1}\wedge\cdots\wedge\widehat{e_{i_j}}\wedge\cdots\wedge e_{i_p}, & \text{if } t=i_j \text{ for some } j,\end{cases}$$

which satisfies $\partial_{\boldsymbol{v}}\iota\zeta+\iota\pi=\mathrm{id}$. Moreover, $\pi\iota=\mathrm{id}$, $\zeta\partial_{\boldsymbol{v}}\iota=\mathrm{id}$. These facts indicate split exactness of the complex. □

Let $\tau^{\geq r}$ be the stupid truncation functor. Since the top row of $\tau^r\mathcal{K}^{\bullet,\bullet}(\boldsymbol{u},\boldsymbol{v};R)$ is the same as $\tau^{\geq 0}(\mathcal{K}^\bullet(\boldsymbol{u};R)[-r])$, we define the morphism $\iota_{t,(r)}$ associated to t as the composition of

$$\tau^{\geq r}(\mathcal{K}^\bullet(\boldsymbol{w};R)[-2r])\xrightarrow{\iota}\tau^{\geq r}(\mathcal{K}^\bullet(\boldsymbol{u};R)[-2r])\hookrightarrow\mathrm{Tot}(\tau^r\mathcal{K}^{\bullet,\bullet}(\boldsymbol{u},\boldsymbol{v};R)).$$

Sometimes we suppress the subscript t in $\iota_{t,(r)}$ if no confusion arises.

Proposition 2.4. *For any $r\geq 0$, $\iota_{(r)}\colon\tau^{\geq r}(\mathcal{K}^\bullet(\boldsymbol{w};R)[-2r])\to\mathrm{Tot}(\tau^r\mathcal{K}^{\bullet,\bullet}(\boldsymbol{u},\boldsymbol{v};R))$ is a quasi-isomorphism with a quasi-inverse $\pi_{(r)}$ induced by π.*

Proof. By Lemmas 2.2, 2.3, the sequence

$$0\longrightarrow\mathcal{K}^\bullet(\boldsymbol{w};R)[1-2r]\xrightarrow{(-1)^\bullet\partial_{\boldsymbol{v}}\iota}\mathcal{K}^\bullet(\boldsymbol{u};R)[-2r]\xrightarrow{\pi}\mathcal{K}^\bullet(\boldsymbol{w};R)[-2r]\longrightarrow 0$$

of cochain complexes is exact. After shifting degrees, we have another exact sequence

$$0\longrightarrow\mathcal{K}^\bullet(\boldsymbol{w};R)[2-2r]\xrightarrow{(-1)^{\bullet-1}\partial_{\boldsymbol{v}}\iota}\mathcal{K}^\bullet(\boldsymbol{u};R)[1-2r]\xrightarrow{\pi}\mathcal{K}^\bullet(\boldsymbol{w};R)[1-2r]\longrightarrow 0.$$

Since $\partial_{\boldsymbol{v}}\iota\zeta+\iota\pi=\mathrm{id}$ (see the proof of Lemma 2.3), we have $(-1)^\bullet\partial_{\boldsymbol{v}}\iota\pi=(-1)^\bullet\partial_{\boldsymbol{v}}(\mathrm{id}-\partial_{\boldsymbol{v}}\iota\zeta)=(-1)^\bullet\partial_{\boldsymbol{v}}$. So the above two exact sequences are combined into a new one

$$0\longrightarrow\mathcal{K}^\bullet(\boldsymbol{w};R)[2-2r]\xrightarrow{(-1)^{\bullet-1}\partial_{\boldsymbol{v}}\iota}\mathcal{K}^\bullet(\boldsymbol{u};R)[1-2r]\xrightarrow{(-1)^\bullet\partial_{\boldsymbol{v}}}\mathcal{K}^\bullet(\boldsymbol{u};R)\xrightarrow{\pi}\mathcal{K}^\bullet(\boldsymbol{w};R)[-2r]\longrightarrow 0.$$

Continuing the procedure, we obtain a long exact sequence

$$\cdots\longrightarrow\mathcal{K}^\bullet(\boldsymbol{u};R)[2-2r]\xrightarrow{(-1)^{\bullet-1}\partial_{\boldsymbol{v}}}\mathcal{K}^\bullet(\boldsymbol{u};R)[1-2r]\xrightarrow{(-1)^\bullet\partial_{\boldsymbol{v}}}\mathcal{K}^\bullet(\boldsymbol{u};R)[-2r]\overset{\pi}{\twoheadrightarrow}\mathcal{K}^\bullet(\boldsymbol{w};R)[-2r].$$

Let the functor $\tau^{\geq r}$ act on the long sequence, and then by using the sign trick, we make all the terms except the last one (i.e. $\tau^{\geq r}(\mathcal{K}^\bullet(\boldsymbol{w};R)[-2r])$) into a double complex. It is obvious that the resulting double complex is nothing but $\tau^r\mathcal{K}^{\bullet,\bullet}(\boldsymbol{u},\boldsymbol{v};R)$. Therefore, π induces a quasi-isomorphism

$$\pi_{(r)}\colon\ \mathrm{Tot}(\tau^r\mathcal{K}^{\bullet,\bullet}(\boldsymbol{u},\boldsymbol{v};R))\longrightarrow\tau^{\geq r}(\mathcal{K}^\bullet(\boldsymbol{w};R)[-2r])$$

which is quasi-inverse to $\iota_{(r)}$. □

Definition 2.1. An n-POS $(\boldsymbol{u},\boldsymbol{v})$ is said to be *proportional* to another one $(\boldsymbol{u}',\boldsymbol{v}')$ if there exist invertible λ, $\mu\in R$ such that $(\boldsymbol{u}',\boldsymbol{v}')=(\lambda\boldsymbol{u},\mu\boldsymbol{v})$.

Notice that the (p,q)-entry of $\tau^r\mathcal{K}^{\bullet,\bullet}(\boldsymbol{u},\boldsymbol{v};R)$ (resp. $\tau^r\mathcal{K}^{\bullet,\bullet}(\boldsymbol{u}',\boldsymbol{v}';R)$) is $\mathcal{K}^{p-q}(\boldsymbol{u},\boldsymbol{v};R)$ (resp. $\mathcal{K}^{p-q}(\boldsymbol{u}',\boldsymbol{v}';R)$), and that $\mathcal{K}^{p-q}(\boldsymbol{u},\boldsymbol{v};R)$ and $\mathcal{K}^{p-q}(\boldsymbol{u}',\boldsymbol{v}';R)$ share the same rank as free R-modules. There are isomorphisms

$$\lambda^p\mu^q\colon \mathcal{K}^{p-q}(\boldsymbol{u},\boldsymbol{v};R)\longrightarrow \mathcal{K}^{p-q}(\boldsymbol{u}',\boldsymbol{v}';R)$$

given by the multiplication by $\lambda^p\mu^q$ for all p, q, and they constitute an isomorphism

$$\xi_{(r)}\colon \tau^r\mathcal{K}^{\bullet,\bullet}(\boldsymbol{u},\boldsymbol{v};R)\longrightarrow \tau^r\mathcal{K}^{\bullet,\bullet}(\boldsymbol{u}',\boldsymbol{v}';R) \tag{2.1}$$

of double complexes. The induced isomorphism between their total complexes is denoted by $\xi_{(r)}^{\mathrm{Tot}}$.

3. Hochschild cohomology of affine hypersurfaces

Let $A=k[y_1,\dots,y_n]/(G)$ be the quotient of the polynomial algebra $k[y_1,\dots,y_n]$ by a unique relation G. There are several papers concerning the Hochschild and cyclic (co)homology of A, the treatment of the topic dating back to Wolffhardt's work on Hochschild homology of (analytic) complete intersections [23]. We base our exposition on the more recent papers [2], [19]. In [19], Michler describes the Hochschild homology groups of A as well as their Hodge decompositions when G is reduced, based on the cotangent complex of A. The Hochschild cohomology groups are not treated in [19]. In [2], the authors from BACH construct a nice finitely generated free resolution $\mathcal{R}^b_\bullet(A)$ of A over A^e under an additional condition on G. For the normalized bar resolution $\bar{C}^{\mathrm{bar}}_\bullet(A)$, the authors give comparison maps

$$\bar{C}^{\mathrm{bar}}_\bullet(A)\underset{\alpha'}{\overset{\alpha}{\rightleftarrows}}\mathcal{R}^b_\bullet(A)$$

satisfying $\alpha\alpha'=\mathrm{id}$. By virtue of the smaller resolution $\mathcal{R}^b_\bullet(A)$, the authors compute the Hochschild homology and cohomology of A.

From now on, we assume that $G=G(y_1,\dots,y_n)$ has leading term y_1^d with respect to the lexicographic ordering $y_1>\cdots>y_n$. Under this assumption, we are able to use the resolution $\mathcal{R}^b_\bullet(A)$ from [2] and obtain the Hochschild (co)homology groups as $H_p(A,A)=H_p(\mathcal{R}_\bullet(A))$ and $H^p(A,A)=H^p(\mathcal{L}^\bullet(A))$ where $\mathcal{R}_\bullet(A)=A\otimes_{A^e}\mathcal{R}^b_\bullet(A)$ and $\mathcal{L}^\bullet(A)=\mathrm{Hom}_{A^e}(\mathcal{R}^b_\bullet(A),A)\cong\mathrm{Hom}_A(\mathcal{R}_\bullet(A),A)$. We also note that $\mathcal{R}_\bullet(A)$ admits a decomposition $\oplus_{r\in\mathbb{N}}\mathcal{R}_\bullet(A)_r$ by the proof of [2, Thm. 3.2.5] and respectively $\mathcal{L}^\bullet(A)$ admits a decomposition $\prod_{r\in\mathbb{N}}\mathcal{L}^\bullet(A)_r$ by the proof of [2, Thm. 3.2.7], and $\mathrm{Hom}_A(\mathcal{R}_\bullet(A)_r,A)\cong\mathcal{L}^\bullet(A)_r$. Moreover, the decomposition $\mathcal{L}^p(A)=\prod_{r\in\mathbb{N}}\mathcal{L}^p(A)_r$ is in fact a finite product for every fixed p. Hence $H^p(A,A)=\oplus_{r\in\mathbb{N}}H^p(\mathcal{L}^\bullet(A)_r)$. In this section, we first make the complex $\mathcal{L}^\bullet(A)$ explicit according to [2], and then restate it in terms of the cotangent complex, inspired by [19]. Next we will prove that the decomposition $H^p(A,A)=\oplus_{r\in\mathbb{N}}H^p(\mathcal{L}^p(A)_r)$ coincides with the Hodge decomposition [8]. Finally, Hochschild cohomology of localizations of A is discussed.

By the construction of [2], denote $\mathcal{L}^\bullet_|(A)=\wedge^\bullet(A\mathfrak{e}_1\oplus\cdots\oplus A\mathfrak{e}_n)$ and then $\mathcal{L}^\bullet(A)$ is the algebra of divided powers over $\mathcal{L}^\bullet_|(A)$ in one variable $\mathfrak{s}$. Set $|\mathfrak{e}_i|=1$ and $|\mathfrak{s}^{(j)}|=2j$, then $\mathcal{L}^\bullet(A)$ is made into a DG A-algebra whose differential is given by $\mathfrak{e}_i\mapsto(\partial G/\partial y_i)\mathfrak{s}^{(1)}$ and $\mathfrak{s}^{(1)}\mapsto 0$. By writing $\mathfrak{e}_{i_1\dots i_l}$ instead of the product $\mathfrak{e}_{i_1}\wedge\cdots\wedge\mathfrak{e}_{i_l}$, we have

$$\mathcal{L}^p(A)=\bigoplus_{\substack{0\le j\le p/2\\ 1\le i_1<\cdots<i_{p-2j}\le n}}A\mathfrak{e}_{i_1\dots i_{p-2j}}\mathfrak{s}^{(j)},$$

and the differential $\mathcal{L}^p(A)\to\mathcal{L}^{p+1}(A)$ is given by

$$\mathfrak{e}_{i_1\dots i_{p-2j}}\mathfrak{s}^{(j)}\longmapsto\sum_{l=1}^{p-2j}(-1)^{l-1}\frac{\partial G}{\partial y_{i_l}}\mathfrak{e}_{i_1\dots\widehat{i_l}\dots i_{p-2j}}\mathfrak{s}^{(j+1)}.$$

It immediately follows that the A-module complex $\mathcal{L}^\bullet(A)$ admits a decomposition $\mathcal{L}^\bullet(A) = \prod_{r\in\mathbb{N}} \mathcal{L}^\bullet(A)_r$ with

$$\mathcal{L}^\bullet(A)_r = \tau^{\geq r}(\mathcal{K}^\bullet((\partial G/\partial y_i)_{1\leq i\leq n}; A)[-2r]). \tag{3.1}$$

Let us shift our attention to the cotangent complex $\mathbb{L}_{A/k}$ of A. As stated in [19] (also see [12, Ch. III, Prop. 3.3.6]), this complex, unique up to homotopy equivalence, is given by

$$0 \longrightarrow Adz \xrightarrow{\delta} \bigoplus_{i=1}^{n} Ady_i \longrightarrow 0$$

where the two nonzero terms sit in degrees -1 and 0 respectively, dz and dy_i are base elements and

$$\delta(dz) = \sum_{i=1}^{n} \frac{\partial G}{\partial y_i} dy_i.$$

By [13, Ch. VIII, Cor. 2.1.2.2], $\wedge^r \mathbb{L}_{A/k}$ is isomorphic to a complex determined by δ in the derived category $\mathbf{D}^b(A)$, more explicitly,

$$\wedge^r \mathbb{L}_{A/k} \cong \bigoplus_{i+j=r} \wedge^i (Ady_1 \oplus \cdots \oplus Ady_n) \otimes_A \mathit{\Gamma}^j(Adz)$$

where $\mathit{\Gamma}^j(-)$ is the degree j component of the divided power functor over A.[1] Taking the dualities on both sides, we obtain

$$\mathrm{Hom}_A(\wedge^r \mathbb{L}_{A/k}, A) \cong \bigoplus_{i+j=r} \wedge^i (A(dy_1)^* \oplus \cdots \oplus A(dy_n)^*) \otimes_A \mathit{\Gamma}^j(A(dz)^*) \tag{3.2}$$

where $(-)^*$ stands for dual basis. Notice that the j-th term of the right-hand side of (3.2) is free of rank $\binom{n}{r-j}$, and the rank is the same as that of $\tau^{\geq 0}(\mathcal{K}^\bullet((\partial G/\partial y_i)_{1\leq i\leq n}; A)[-r])$ for all $0 \leq j \leq r$. By taking into account differentials, one has an isomorphism $\mathrm{Hom}_A(\wedge^r \mathbb{L}_{A/k}, A) \cong \tau^{\geq 0}(\mathcal{K}^\bullet((\partial G/\partial y_i)_{1\leq i\leq n}; A)[-r])$, and further $\mathrm{Hom}_A(\wedge^r \mathbb{L}_{A/k}, A)[-r] \cong \mathcal{L}^\bullet(A)_r$ by (3.1). Consequently we have

$$H^p(A,A) = \bigoplus_{r\in\mathbb{N}} H^p(\mathcal{L}^\bullet(A)_r) \cong \bigoplus_{r\in\mathbb{N}} H^p(\mathrm{Hom}_A(\wedge^r \mathbb{L}_{A/k}, A)[-r]) = \bigoplus_{r\in\mathbb{N}} \mathrm{Ext}_A^{p-r}(\wedge^r \mathbb{L}_{A/k}, A).$$

We will compare the above formula with the Hodge decomposition $H^p(A,A) = \oplus_{r\in\mathbb{N}} H^p_{(r)}(A,A)$. To this end, let us observe that $H_p(A,A) = H_p(\oplus_{r\in\mathbb{N}} \mathcal{R}_\bullet(A)_r) \cong \oplus_{r\in\mathbb{N}} H_p(\mathcal{R}_\bullet(A)_r)$. The direct summand $H_p(\mathcal{R}_\bullet(A)_r)$ is isomorphic to the Hodge component $H_p^{(r)}(A,A)$ by [19]. This immediately implies that both quasi-isomorphisms $\mathrm{id}\otimes\alpha$, $\mathrm{id}\otimes\alpha'$ in

$$\bigoplus_{r\in\mathbb{N}} \bar{C}_\bullet(A,A)_r \underset{\mathrm{id}\otimes\alpha'}{\overset{\mathrm{id}\otimes\alpha}{\rightleftarrows}} \bigoplus_{r\in\mathbb{N}} \mathcal{R}_\bullet(A)_r$$

can be replaced by another pair of quasi-isomorphisms $\tilde{\alpha}$, $\tilde{\alpha}'$ which preserve the above direct sums. In fact, if $p \geq 1$ then $\mathrm{id}\otimes\alpha_p \colon \oplus_{r\in\mathbb{N}} \bar{C}_p(A,A)_r \to \oplus_{r\in\mathbb{N}} \mathcal{R}_p(A)_r$ can be represented by a matrix $(a_{ij})_{p\times p}$ since $\bar{C}_p(A,A)_r$ and $\mathcal{R}_p(A)_r$ are zero unless $1 \leq r \leq p$. Let $\tilde{\alpha}_p$ be represented by the matrix $\mathrm{diag}(a_{11}, a_{22}, \ldots, a_{pp})$. If $p = 0$, let $\tilde{\alpha}_0 = \mathrm{id}\otimes\alpha_0$. Then $\tilde{\alpha}$ is a quasi-isomorphism, as desired. Similar matrix construction holds for $\tilde{\alpha}'$.

By applying $\mathrm{Hom}_A(-,A)$, we get quasi-isomorphisms

$$\prod_{r\in\mathbb{N}} \bar{C}^\bullet(A,A)_r \underset{\beta:=\mathrm{Hom}(\tilde{\alpha},A)}{\overset{\beta':=\mathrm{Hom}(\tilde{\alpha}',A)}{\rightleftarrows}} \prod_{r\in\mathbb{N}} \mathcal{L}^\bullet(A)_r$$

which preserve direct products. Taking cohomology, we then obtain

$$H^p_{(r)}(A,A) = H^p(\mathcal{L}^\bullet(A)_r) \cong \mathrm{Ext}_A^{p-r}(\wedge^r \mathbb{L}_{A/k}, A).$$

[1] Upright $\Gamma(X,-)$ will denote the global section functor on a scheme X in §4.

From it, we know the decomposition of $H^p(\mathcal{L}^\bullet(A))$ deduced from [2] actually corresponds to the Hodge decomposition.

Observe that the quasi-isomorphism $\beta\colon \mathcal{L}^\bullet(A) \to \bar{C}^\bullet(A,A)$ induces isomorphisms $H^p_{(r)}(A,A) \cong H^p(\mathcal{L}^\bullet(A)_r)$ for all p, r. The explicit expression of β can be concluded from α, and we will give it later on. For our purpose, we first introduce some cochains. Note that the algebra A has the basis

$$\mathcal{B}_A = \{y_1^{p_1} y_2^{p_2} \cdots y_n^{p_n} \mid 0 \le p_1 \le d-1,\, p_2, \ldots, p_n \in \mathbb{N}\}.$$

We define for $1 \le l \le n$ a normalized 1-cochain ${}^\circ\partial/\partial y_l$ by

$$\mathcal{B}_A \ni y_1^{p_1} y_2^{p_2} \cdots y_n^{p_n} = f \longmapsto \frac{{}^\circ\partial f}{\partial y_l} = p_l y_1^{p_1} \cdots y_{l-1}^{p_{l-1}} y_l^{p_l-1} y_{l+1}^{p_{l+1}} \cdots y_n^{p_n} \tag{3.3}$$

and a normalized 2-cochain ${}^\circ\mu$ by

$${}^\circ\mu(f,g) = \begin{cases} 0, & p_1+q_1<d, \\ y_1^{p_1+q_1-d} y_2^{p_2+q_2} \cdots y_n^{p_n+q_n}, & p_1+q_1 \ge d. \end{cases} \tag{3.4}$$

for an additional $g = y_1^{q_1} y_2^{q_2} \cdots y_n^{q_n} \in \mathcal{B}_A$. One can easily check that ${}^\circ\mu$ is a 2-cocycle.

Now we give the expression of $\beta = \sum_r \beta_{(r)}\colon \mathcal{L}^\bullet(A) \to \bar{C}^\bullet(A,A)$:

$$\beta_{(p-j)}(\mathfrak{e}_{i_1 \ldots i_{p-2j}} \mathfrak{s}^{(j)}) = (-1)^{\binom{p-2j}{2}} \frac{{}^\circ\partial}{\partial y_{i_1}} \cup \cdots \cup \frac{{}^\circ\partial}{\partial y_{i_{p-2j}}} \cup {}^\circ\mu^{\cup j}. \tag{3.5}$$

The notation $\cup$, not to be confused with the well-known cup product, is defined as

$$P_1 \cup P_2 \cup \cdots \cup P_m = \frac{1}{m!} \sum_{\sigma \in S_m} (-1)^c \mu \circ \left(P_{\sigma^{-1}(1)} \otimes P_{\sigma^{-1}(2)} \otimes \cdots \otimes P_{\sigma^{-1}(m)}\right)$$

where $P_i \in \bar{C}^\bullet(A,A)$, μ is the multiplication map (or rather its unique extension by associativity to an m-ary multiplication map) and

$$c = \#\{(i,j) \mid i<j,\, \sigma^{-1}(i) > \sigma^{-1}(j),\, P_i, P_j \text{ have odd degrees}\}.$$

Thus, the operation $\cup$ becomes supercommutative. For example,

$$\frac{{}^\circ\partial}{\partial y_i} \cup {}^\circ\mu = \frac{1}{2}\mu \circ \left(\frac{{}^\circ\partial}{\partial y_i} \otimes {}^\circ\mu + {}^\circ\mu \otimes \frac{{}^\circ\partial}{\partial y_i}\right) = {}^\circ\mu \cup \frac{{}^\circ\partial}{\partial y_i},$$
$$\frac{{}^\circ\partial}{\partial y_i} \cup \frac{{}^\circ\partial}{\partial y_j} = -\frac{{}^\circ\partial}{\partial y_j} \cup \frac{{}^\circ\partial}{\partial y_i}.$$

Remark 3.1. Since $\beta_{(r)}(\mathcal{L}^\bullet(A)_r) \subseteq \bar{C}^\bullet(A,A)_r$, we also call $\mathcal{L}^\bullet(A) = \oplus_{r\in\mathbb{N}} \mathcal{L}^\bullet(A)_r$ the *Hodge decomposition.*

Remark 3.2. Recall that the vanishing of the groups $H^2_{(1)}(A,M)$ for all A-modules M characterizes smoothness of A. Since A is an affine hypersurface, A is smooth if and only if the ideal $(\partial G/\partial y_1, \ldots, \partial G/\partial y_n)$ is equal to A itself.

Let $\bar{A}$ be the localization of A at a multiplicatively closed set generated by $y_{t_1}, \ldots, y_{t_h}$ where $2 \le t_1 < \cdots < t_h \le n$. Let $\sigma\colon \bar{A} \to B$ be a morphism of commutative algebras such that B is a flat $\bar{A}$-module via σ. Then $\bar{A}$ has a basis

$$\mathcal{B}_{\bar{A}} = \{y_1^{p_1} y_2^{p_2} \cdots y_n^{p_n} \mid 0 \le p_1 \le d-1, p_{t_1}, \ldots, p_{t_h} \in \mathbb{Z},\ \text{other } p_i \in \mathbb{N}\}.$$

As above, cochains ${}^\circ\partial/\partial y_l \in \bar{C}^1(\bar{A},\bar{A})$ and ${}^\circ\mu \in \bar{C}^2(\bar{A},\bar{A})$ can be defined similarly. After composing them with σ, we obtain cochains in $\bar{C}^1(\bar{A},B)$, $\bar{C}^2(\bar{A},B)$. Furthermore, one can easily check that there is a quasi-isomorphism $\beta\colon B \otimes_A \mathcal{L}^\bullet(A) \to \bar{C}^\bullet(\bar{A},B)$ whose expression is similar to the one shown in (3.5).

4. Hochschild cohomology of projective hypersurfaces

For any morphism $X \to Y$ of schemes or analytic spaces, Buchweitz and Flenner introduce the Hochschild complex $\mathbb{H}_{X/Y}$ of X over Y [5], and they deduce an isomorphism $\mathbb{H}_{X/Y} \cong \mathbb{S}(\mathbb{L}_{X/Y}[1])$ in the derived category $\mathbf{D}(X)$ where $\mathbb{L}_{X/Y}$ denotes the cotangent complex of X over Y and $\mathbb{S}(\mathbb{L}_{X/Y}[1])$ is the derived symmetric algebra [4]. As a consequence, there is a decomposition of Hochschild cohomology in terms of the derived exterior powers of the cotangent complex

$$HH^i(X) \cong \bigoplus_{p+q=i} \mathrm{Ext}^p_X(\wedge^q \mathbb{L}_{X/k}, \mathcal{O}_X) \tag{4.1}$$

in the special case $Y = \operatorname{Spec} k$, which generalizes the HKR decomposition in the smooth case. Around the same time, Schuhmacher also deduced the decomposition (4.1) using a different method [21].

This decomposition is more computable than using Gerstenhaber-Schack complex directly. However, we do not use it for our computation since its deformation behavior is implicit. As a sequel to [6], [17], we compute $HH^i(X)$ starting from the Gerstenhaber-Schack complex, since a deformation interpretation of Gerstenhaber-Schack 2-cocycles is at hand [6]. In §4.1 we construct a series of complexes of $\mathcal{O}_X$-modules, whose associated simplicial complexes $\mathcal{E}_r^\bullet$ are much smaller than the Hodge components $\bar{\mathbf{C}}'_{\mathrm{GS}}(\mathcal{O}_X|_{\mathfrak{V}})_r$ of the normalized reduced Gerstenhaber-Schack complex (for a chosen covering $\mathfrak{V}$). Using the technique from §2, we construct explicit quasi-isomorphisms $\mathcal{E}_r^\bullet \to \bar{\mathbf{C}}'_{\mathrm{GS}}(\mathcal{O}_X|_{\mathfrak{V}})_r$ for all r. Hence the Hodge decomposition (1.3) of $HH^\bullet(X)$ is obtained.

Due to the theoretical significance of the cotangent complex, we give expressions of $\wedge^r \mathbb{L}_{X/k}$ in terms of twisted structure sheaves $\mathcal{O}_X(l)$ for all r in §4.2 when X is a projective hypersurface. This allows us to explain directly how our results agree with Buchweitz and Flenner's. In particular, the decompositions (4.1) and (1.3) agree for any projective hypersurface.

4.1. Double complexes and quasi-isomorphisms.

Let $n \geq 2$, $R = k[x_0, \ldots, x_n]$ and $F \in R$ be a homogeneous polynomial of degree $d \geq 2$. Let $S = R/(F)$ and $X = \operatorname{Proj} S \subset \mathbb{P}^n$. Choose a point in $\mathbb{P}^n$ where F does not vanish, and change variables so that this point is $(1 : 0 : \cdots : 0)$; then the coefficient of x_0^d does not vanish. There is no harm to assume that the coefficient is equal to one. In this way, X can be covered by the standard covering

$$\mathfrak{U} = \{U_i = X \cap \{x_i \neq 0\} \mid 1 \leq i \leq n\}.$$

Let $\mathfrak{V} = \{V_{i_1 \ldots i_s} = U_{i_1} \cap \cdots \cap U_{i_s} \mid 1 \leq i_1 < \cdots < i_s \leq n\}$ be the associated covering closed under intersections. For any a p-simplex $\sigma \in \mathcal{N}_p(\mathfrak{V})$, say

$$\sigma = (V_0 \subseteq V_1 \subseteq \cdots \subseteq V_p),$$

denote its domain V_0 and codomain V_p by ${}_\diamond\sigma$ and $\sigma_\diamond$ respectively. Let $\mathbf{C}^{\bullet,\bullet}(\mathcal{A})$ be the Gerstenhaber-Schack double complex where $\mathcal{A} = \mathcal{O}_X|_{\mathfrak{V}}$, namely,

$$\mathbf{C}^{p,q}(\mathcal{A}) = \prod_{\sigma \in \mathcal{N}_p(\mathfrak{V})} \mathrm{Hom}_k(\mathcal{A}(\sigma_\diamond)^{\otimes q}, \mathcal{A}({}_\diamond\sigma))$$

endowed with the (vertical) product Hochschild differential d_{Hoch} and the (horizontal) simplicial differential d_{simp}. Recall that a cochain $f = (f_\sigma) \in \mathbf{C}^{p,q}(\mathcal{A})$ is called normalized if for any p-simplex σ, f_σ is normalized, and it is called reduced if $f_\sigma = 0$ whenever σ is degenerate. Let $\bar{\mathbf{C}}'^{\bullet,\bullet}(\mathcal{A})$ be the normalized reduced sub-double complex of $\mathbf{C}^{\bullet,\bullet}(\mathcal{A})$ and $\bar{\mathbf{C}}'^{\bullet}_{\mathrm{GS}}(\mathcal{A})$ be the associated total complex.

Observe that for $1 \le i \le n$, $A_i = \mathcal{A}(U_i) = k[y_0, \ldots, \widehat{y_i}, \ldots, y_n]/(G_i)$ where

$$G_i = F(y_0, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n) = y_0^d + \cdots$$

is monic. Here we assign an ordering $y_0 > \cdots > y_{i-1} > y_{i+1} > \cdots > y_n$. So we have complexes $\mathcal{L}^\bullet(A_i)$ as given in §3. Denote by $\boldsymbol{w}_i$ the sequence

$$\left(\frac{\partial G_i}{\partial y_0}, \ldots, \frac{\partial G_i}{\partial y_{i-1}}, \frac{\partial G_i}{\partial y_{i+1}}, \ldots, \frac{\partial G_i}{\partial y_n}\right).$$

Then $\mathcal{L}^\bullet(A_i)_r = \tau^{\ge r}(\mathcal{K}^\bullet(\boldsymbol{w}_i; A_i)[-2r])$.

For any $V \in \mathfrak{V}$, say $V = V_{i_1 \ldots i_s}$, let $\Phi(V) = \{i_1, \ldots, i_s\}$. If $t \in \Phi(V)$, we may express $\mathcal{A}(V)$ in term of generators and relations as

$$\mathcal{A}(V,t) = k[y_0, \ldots, \widehat{y_t}, \ldots, y_n, y_{t_1}^{-1}, \ldots, \widehat{y_t^{-1}}, \ldots, y_{t_s}^{-1}]/(G_t, y_{t_1}y_{t_1}^{-1} - 1, \ldots, y_{t_s}y_{t_s}^{-1} - 1).$$

Since $\mathcal{A}(V,t)$ is a localization of A_t, there is a quasi-isomorphism

$$\beta \colon B \otimes_{A_t} \mathcal{L}^\bullet(A_t) \longrightarrow \bar{C}^\bullet(\mathcal{A}(V,t), B)$$

for any flat morphism $\mathcal{A}(V,t) \to B$ by the last paragraph of §3. If s also belongs to $\Phi(V)$, the canonical isomorphism $\mathcal{A}(V,t) \to \mathcal{A}(V,s)$ is denoted by $\zeta_{t,s}$. Unfortunately, $\zeta_{t,s}$ is not compatible with the differentials of $\mathcal{L}^\bullet(A_t)$ and $\mathcal{L}^\bullet(A_s)$, namely, the square

$$\begin{array}{ccc} B \otimes_{A_t} \mathcal{L}^\bullet(A_t) & \xrightarrow{\beta} & \bar{C}^\bullet(\mathcal{A}(V,t), B) \\ \downarrow{\scriptstyle \zeta_{t,s}} & & \uparrow{\scriptstyle \zeta_{t,s}^*} \\ B \otimes_{A_s} \mathcal{L}^\bullet(A_s) & \xrightarrow{\beta} & \bar{C}^\bullet(\mathcal{A}(V,s), B) \end{array}$$

fails to be commutative. So one does not expect that the complexes $\mathcal{L}^\bullet(\mathcal{A}(V))$ for all affine pieces V can be made into a complex $\mathcal{L}^\bullet$ of sheaves on X equipped with nice restriction maps. The reason is that the $\mathcal{L}^\bullet(\mathcal{A}(V))$'s are too small. In order to study $\mathcal{A}$ globally, we have to put on their weight so that our computation will be easier.

It follows from Euler's formula

$$\sum_{i=0}^{n} \frac{\partial F}{\partial x_i} \cdot x_i = d \cdot F$$

that

$$\boldsymbol{u} = \left(\frac{\partial F}{\partial x_0}, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\right) \text{ and } \boldsymbol{v} = (x_0, x_1, \ldots, x_n)$$

make up an n-POS in S. Also, there is an n-POS $(\boldsymbol{u}_i, \boldsymbol{v}_i)$ in A_i:

$$\boldsymbol{u}_i = \left(\frac{\partial G_i}{\partial y_0}, \ldots, \frac{\partial G_i}{\partial y_{i-1}}, H_i, \frac{\partial G_i}{\partial y_{i+1}}, \ldots, \frac{\partial G_i}{\partial y_n}\right) \text{ and } \boldsymbol{v}_i = (y_0, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n)$$

where

$$H_i = \frac{\partial F}{\partial x_i}(y_0, y_1, \ldots, y_{i-1}, 1, y_{i+1}, \ldots, y_n).$$

Since $\boldsymbol{w}_i$ is the subsequence of $\boldsymbol{u}_i$ by deleting H_i, the results from §2 apply. As before we get the mixed complex $\mathcal{K}^\bullet(\boldsymbol{u}, \boldsymbol{v}; S)$ and the double complex $\mathcal{K}^{\bullet,\bullet}(\boldsymbol{u}, \boldsymbol{v}; S)$.

Let $r \ge 0$ and let us consider $\tau^r \mathcal{K}^{\bullet,\bullet}(\boldsymbol{u}, \boldsymbol{v}; S)$. We twist the degrees of its entries as in Figure 2 so that it is made into a double complex of graded S-modules. The associated total complex gives rise to a complex of sheaves

$$\mathcal{F}_r^\bullet: \quad \mathcal{O}_X \longrightarrow \mathcal{O}_X(1)^{n+1} \longrightarrow \cdots \longrightarrow \mathcal{O}_X(rd - d + 1)^{n+1} \longrightarrow \mathcal{O}_X(rd).$$

We in turn have double complexes $\mathcal{E}_r^{\bullet,\bullet}$, $\mathcal{G}_r^{\bullet,\bullet}$ and $\mathcal{H}_r^{\bullet,\bullet}$ as follows:

$$\mathcal{E}_r^{p,q} = \mathbf{C}^p_{\text{simp}}(\mathfrak{V}, \mathcal{F}_r^q|_{\mathfrak{V}}) = \prod_{\sigma \in \mathcal{N}_p(\mathfrak{V})} \mathcal{F}_r^q(\diamond\sigma),$$

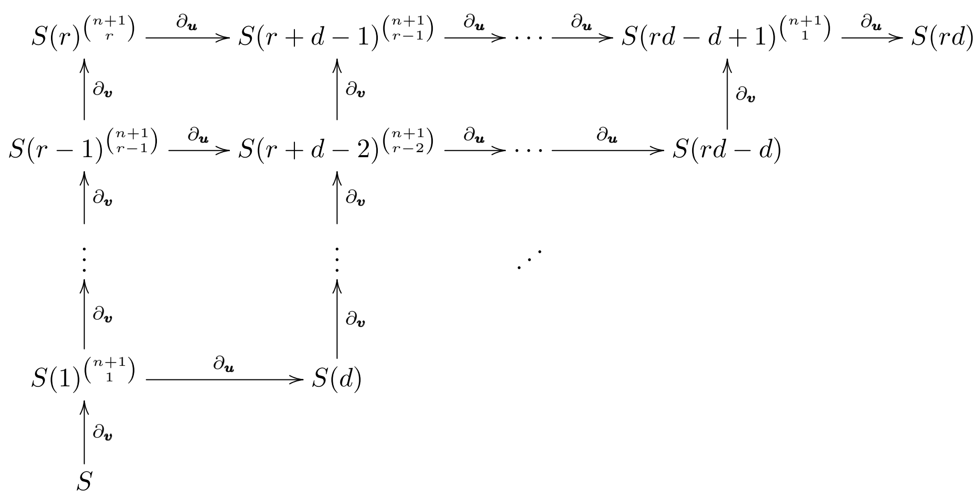

FIGURE 2. Double complex $\tau^r\mathcal{K}^{\bullet,\bullet}(\boldsymbol{u},\boldsymbol{v};S)$

$$\mathcal{G}_r^{p,q} = \check{\mathbf{C}}^p(\mathfrak{V},\mathcal{F}_r^q) = \prod_{V_{i_1}\dots V_{i_p}} \mathcal{F}_r^q(V_{i_1}\cap\cdots\cap V_{i_p}),$$
$$\mathcal{H}_r^{p,q} = \check{\mathbf{C}}^p(\mathfrak{U},\mathcal{F}_r^q) = \prod_{U_{i_1}\dots U_{i_p}} \mathcal{F}_r^q(U_{i_1}\cap\cdots\cap U_{i_p}).$$

Their associated complexes are denoted by $\mathcal{E}_r^\bullet$, $\mathcal{G}_r^\bullet$ and $\mathcal{H}_r^\bullet$ respectively.

Lemma 4.1. *There exist morphisms $\mathcal{H}_r^{\bullet,\bullet}\to\mathcal{G}_r^{\bullet,\bullet}\to\mathcal{E}_r^{\bullet,\bullet}$ which induce quasi-isomorphisms $\mathcal{H}_r^\bullet\to\mathcal{G}_r^\bullet\to\mathcal{E}_r^\bullet$ for all r.*

Proof. The existence of $\mathcal{H}_r^{\bullet,\bullet}\to\mathcal{G}_r^{\bullet,\bullet}$ is clear since $\mathfrak{V}$ is a refinement of $\mathfrak{U}$. The morphism $\mathcal{G}_r^{\bullet,\bullet}\to\mathcal{E}_r^{\bullet,\bullet}$ is given in [6]. Both induce quasi-isomorphisms of their total complexes, by using the spectral sequence argument. □

For the purpose of studying deformations in the following sections, let us make the composition explicit. Fix a map $\lambda\colon\mathfrak{V}\to\mathfrak{U}$ such that $V\subseteq\lambda(V)$ for all $V\in\mathfrak{V}$. The induced quasi-isomorphism $\bar\lambda\colon\mathcal{H}_r^\bullet\to\mathcal{E}_r^\bullet$ maps $f\in\mathcal{H}_r^{p,q}$ to

$$\bar\lambda(f)_{V_{j_0}\subseteq\cdots\subseteq V_{j_p}} = f_{\lambda(V_{j_0})\dots\lambda(V_{j_p})}. \tag{4.2}$$

Let $\bar{\mathbf{C}}'^{\bullet,\bullet}(\mathcal{A}) = \oplus_{r\in\mathbb{N}}\bar{\mathbf{C}}'^{\bullet,\bullet}_r(\mathcal{A})$ be the Hodge decomposition. Our goal is to construct a family of morphisms $\mathcal{E}_r^{\bullet,\bullet}\to\bar{\mathbf{C}}'^{\bullet,\bullet}_r(\mathcal{A})$ of double complexes for all r that give rise to quasi-isomorphisms $\mathcal{E}_r^\bullet\to\bar{\mathbf{C}}'^\bullet_{\mathrm{GS}}(\mathcal{A})_r$. Since the cohomology of $\bar{\mathbf{C}}'^\bullet_{\mathrm{GS}}(\mathcal{A})$ turns out to be isomorphic to the Hochschild cohomology of X (see [17, Thm. 7.8.1]), the cohomology $HH^\bullet(X)$ can be computed by $\mathcal{H}^\bullet := \oplus_{r\in\mathbb{N}}\mathcal{H}_r^\bullet$, namely, $HH^i(X)\cong H^i(\mathcal{H}^\bullet)$.

Let $\sigma\in\mathcal{N}_p(\mathfrak{V})$ be a p-simplex and consider $t,\ s\in\Phi(\sigma_\diamond)$. We have quasi-isomorphisms

$$\beta_t\colon \bigoplus_{r\in\mathbb{N}}\tau^{\geq r}(\mathcal{K}^\bullet(\boldsymbol{w}_t;\mathcal{A}({}_\diamond\sigma,t))[-2r]) \cong \mathcal{A}({}_\diamond\sigma,t)\otimes_{A_t}\mathcal{L}^\bullet(A_t) \longrightarrow \bar{C}^\bullet(\mathcal{A}(\sigma_\diamond,t),\mathcal{A}({}_\diamond\sigma,t))$$

and β_s, which is defined similarly. Let ${}^\circ\partial_t/\partial y_i$, ${}^\circ\mu_t$ and ${}^\circ\partial_s/\partial y_i$, ${}^\circ\mu_s$ be the resulting Hochschild cochains as defined in (3.3) and (3.4). According to the generators and relations of $\mathcal{A}(\sigma_\diamond,t)$ and $\mathcal{A}({}_\diamond\sigma,t)$, we can regard ${}^\circ\partial_t/\partial y_i$, ${}^\circ\mu_t$ to be cochains in $\bar{C}^\bullet(\mathcal{A}(\sigma_\diamond,t),\mathcal{A}({}_\diamond\sigma,t))$ by abuse of notation, and similarly for ${}^\circ\partial_s/\partial y_i$, ${}^\circ\mu_s$.

Lemma 4.2. *Let $\zeta'_{t,s}\colon\bar{C}^\bullet(\mathcal{A}(\sigma_\diamond,t),\mathcal{A}({}_\diamond\sigma,t))\to\bar{C}^\bullet(\mathcal{A}(\sigma_\diamond,s),\mathcal{A}({}_\diamond\sigma,s))$ be the isomorphism induced by $\zeta_{t,s}$. Then*

(1) $\zeta'_{t,s}(^\circ\partial_t/\partial y_i) = y_t \cdot {}^\circ\partial_s/\partial y_i$ *if* $i \neq t,\ s$.
(2) $\zeta'_{t,s}(^\circ\partial_t/\partial y_s) = -\sum_{i\neq s} y_t y_i \cdot {}^\circ\partial_s/\partial y_i$.
(3) $\zeta'_{t,s}(^\circ\mu_t) = y_t^d \cdot {}^\circ\mu_s$.

Proof. (1) (2) Choose any $f = y_0^{p_0}\cdots y_{s-1}^{p_{s-1}}y_{s+1}^{p_{s+1}}\cdots y_n^{p_n} \in \mathcal{B}_{\mathcal{A}(\sigma_\diamond,s)}$ and let $|f| = \sum_{i\neq t,s} p_i$. We have

$$\begin{aligned}\zeta'_{t,s}\left(\frac{^\circ\partial_t}{\partial y_i}\right)(f) &= \zeta_{t,s}\circ\frac{^\circ\partial_t}{\partial y_i}\circ\zeta_{s,t}(f)\\ &= \zeta_{t,s}\circ\frac{^\circ\partial_t}{\partial y_i}(y_0^{p_0}\cdots y_{t-1}^{p_{t-1}}y_{t+1}^{p_{t+1}}\cdots y_s^{-|f|}\cdots y_n^{p_n})\\ &= \zeta_{t,s}(p_i y_0^{p_0}\cdots y_i^{p_i-1}\cdots y_{t-1}^{p_{t-1}}y_{t+1}^{p_{t+1}}\cdots y_s^{-|f|}\cdots y_n^{p_n})\\ &= p_i y_0^{p_0}\cdots y_i^{p_i-1}\cdots y_t^{p_t+1}\cdots y_{s-1}^{p_{s-1}}y_{s+1}^{p_{s+1}}\cdots y_n^{p_n}\\ &= y_t\frac{^\circ\partial_s}{\partial y_i}(f)\end{aligned}$$

for all $i \neq t,\ s$, and

$$\begin{aligned}\zeta'_{t,s}\left(\frac{^\circ\partial_t}{\partial y_s}\right)(f) &= \zeta_{t,s}\circ\frac{^\circ\partial_t}{\partial y_s}(y_0^{p_0}\cdots y_{t-1}^{p_{t-1}}y_{t+1}^{p_{t+1}}\cdots y_s^{-|f|}\cdots y_n^{p_n})\\ &= \zeta_{t,s}(-|f|y_0^{p_0}\cdots y_{t-1}^{p_{t-1}}y_{t+1}^{p_{t+1}}\cdots y_s^{-|f|-1}\cdots y_n^{p_n})\\ &= -|f|y_0^{p_0}\cdots y_t^{p_t+1}\cdots y_{s-1}^{p_{s-1}}y_{s+1}^{p_{s+1}}\cdots y_n^{p_n}\\ &= -y_t|f|f\\ &= -\sum_{i\neq s} y_t y_i\frac{^\circ\partial_s}{\partial y_i}(f).\end{aligned}$$

(3) Let $g = y_0^{q_0}\cdots y_{s-1}^{q_{s-1}}y_{s+1}^{q_{s+1}}\cdots y_n^{q_n} \in \mathcal{B}_{\mathcal{A}(\sigma_\diamond,s)}$ and $|g| = \sum_{i\neq t,s} q_i$. Assume $p_0 + q_0 \geq d$. Then

$$\begin{aligned}\zeta'_{t,s}(^\circ\mu_t)(f,g) &= \zeta_{t,s}\circ{}^\circ\mu_t(y_0^{p_0}\cdots y_{t-1}^{p_{t-1}}y_{t+1}^{p_{t+1}}\cdots y_s^{-|f|}\cdots y_n^{p_n}, y_0^{q_0}\cdots y_{t-1}^{q_{t-1}}y_{t+1}^{q_{t+1}}\cdots y_s^{-|g|}\cdots y_n^{q_n})\\ &= \zeta_{t,s}(y_0^{p_0+q_0-d}\cdots y_{t-1}^{p_{t-1}+q_{t-1}}y_{t+1}^{p_{t+1}+q_{t+1}}\cdots y_s^{-|f|-|g|}\cdots y_n^{p_n+q_n})\\ &= y_0^{p_0+q_0-d}\cdots y_t^{p_t+q_t+d}\cdots y_{s-1}^{p_{s-1}+q_{s-1}}y_{s+1}^{p_{s+1}+q_{s+1}}\cdots y_n^{p_n+q_n}\\ &= y_t^d\cdot{}^\circ\mu_s(f,g).\end{aligned}$$

On the other hand, $\zeta'_{t,s}(^\circ\mu_t)(f,g) = 0 = y_t^d \cdot {}^\circ\mu_s(f,g)$ trivially holds if $p_0 + q_0 < d$. □

There are proportional n-POS $(\zeta_{t,s}(\boldsymbol{u}_t), \zeta_{t,s}(\boldsymbol{v}_t))$, $(\boldsymbol{u}_s, \boldsymbol{v}_s)$ in $\mathcal{A}(_\diamond\sigma, s)$ with $\boldsymbol{u}_s = y_t^{d-1}\zeta_{t,s}(\boldsymbol{u}_t)$ and $\boldsymbol{v}_s = y_t\zeta_{t,s}(\boldsymbol{v}_t)$. There is an isomorphism

$$\xi^{\mathrm{Tot}}_{t,s,(r)}\colon\ \mathrm{Tot}(\tau^r\mathcal{K}^{\bullet,\bullet}(\zeta_{t,s}(\boldsymbol{u}_t),\zeta_{t,s}(\boldsymbol{v}_t);\mathcal{A}(_\diamond\sigma,s))) \longrightarrow \mathrm{Tot}(\tau^r\mathcal{K}^{\bullet,\bullet}(\boldsymbol{u}_s,\boldsymbol{v}_s;\mathcal{A}(_\diamond\sigma,s)))$$

as given in (2.1). Since the t-th, s-th components of $\zeta_{t,s}(\boldsymbol{v}_t)$ and $\boldsymbol{v}_s$ are invertible, we have the diagram

$$(4.3)\qquad \begin{array}{ccc}\mathrm{Tot}(\tau^r\mathcal{K}^{\bullet,\bullet}(\zeta_{t,s}(\boldsymbol{u}_t),\zeta_{t,s}(\boldsymbol{v}_t);\mathcal{A}(_\diamond\sigma,s))) & \xrightarrow{\pi_{t,(r)}} & \tau^{\geq r}(\mathcal{K}^\bullet(\zeta_{t,s}(\boldsymbol{w}_t);\mathcal{A}(_\diamond\sigma,s))[-2r])\\ & & \downarrow{\scriptstyle \beta'_{t,(r)}}\\ \downarrow{\scriptstyle \xi^{\mathrm{Tot}}_{t,s,(r)}} & & \bar{C}^\bullet(\mathcal{A}(\sigma_\diamond,s),\mathcal{A}(_\diamond\sigma,s))\\ & & \uparrow{\scriptstyle \beta_{s,(r)}}\\ \mathrm{Tot}(\tau^r\mathcal{K}^{\bullet,\bullet}(\boldsymbol{u}_s,\boldsymbol{v}_s;\mathcal{A}(_\diamond\sigma,s))) & \xrightarrow{\pi_{s,(r)}} & \tau^{\geq r}(\mathcal{K}^\bullet(\boldsymbol{w}_s;\mathcal{A}(_\diamond\sigma,s))[-2r])\end{array}$$

where $\beta'_{t,(r)}$ is induced by $\beta_{t,(r)}$ and $\zeta_{t,s}$.

Lemma 4.3. *The diagram* (4.3) *is commutative.*

Proof. Choose any base element $E = e_{i_1} \wedge \cdots \wedge e_{i_p} \in \mathcal{K}^p(\zeta_{t,s}(\boldsymbol{u}_t), \zeta_{t,s}(\boldsymbol{v}_t); \mathcal{A}(\diamond\sigma, s))$. When viewed as a cochain in $\tau^r\mathcal{K}^{\bullet,\bullet}(\zeta_{t,s}(\boldsymbol{u}_t), \zeta_{t,s}(\boldsymbol{v}_t); \mathcal{A}(\diamond\sigma, s))$, E locates in position $(r-p, r)$. So $\xi^{\mathrm{Tot}}_{t,s,(r)}(E) = (y_t^{d-1})^{r-p} y_t^r E = y_t^{r+(d-1)(r-p)} E$. Let us prove the lemma by a case-by-case argument.

If t, $s \notin \{i_1, \ldots, i_p\}$, then

$$\begin{aligned}
\beta'_{t,(r)} \circ \pi_{t,(r)}(E) &= \beta'_{t,(r)}(\mathfrak{e}_{i_1\ldots i_p}\mathfrak{s}^{(r-p)}) \\
&= \zeta_{t,s} \circ (-1)^{\binom{p}{2}} \left(\frac{{}^\circ\partial_t}{\partial y_{i_1}} \cup \cdots \cup \frac{{}^\circ\partial_t}{\partial y_{i_p}} \cup {}^\circ\mu_t^{\cup(r-p)} \right) \circ (\zeta_{s,t})^{\otimes(2r-p)} \\
&= (-1)^{\binom{p}{2}} \zeta'_{t,s}\left(\frac{{}^\circ\partial_t}{\partial y_{i_1}} \right) \cup \cdots \cup \zeta'_{t,s}\left(\frac{{}^\circ\partial_t}{\partial y_{i_p}} \right) \cup \left(\zeta'_{t,s}({}^\circ\mu_t)\right)^{\cup(r-p)} \\
&= (-1)^{\binom{p}{2}} y_t \frac{{}^\circ\partial_s}{\partial y_{i_1}} \cup \cdots \cup y_t \frac{{}^\circ\partial_s}{\partial y_{i_p}} \cup (y_t^d \cdot {}^\circ\mu_s)^{\cup(r-p)} \\
&= y_t^{r+(d-1)(r-p)} (-1)^{\binom{p}{2}} \frac{{}^\circ\partial_s}{\partial y_{i_1}} \cup \cdots \cup \frac{{}^\circ\partial_s}{\partial y_{i_p}} \cup {}^\circ\mu_s^{\cup(r-p)} \\
&= y_t^{r+(d-1)(r-p)} \beta_{s,(r)}(\mathfrak{e}_{i_1\ldots i_p}\mathfrak{s}^{(r-p)}) \\
&= \beta_{s,(r)} \circ \pi_{s,(r)} \circ \xi^{\mathrm{Tot}}_{t,s,(r)}(E).
\end{aligned}$$

If $s \notin \{i_1, \ldots, i_p\}$ and $t = i_j$ for some j, then

$$\begin{aligned}
\beta'_{t,(r)} \circ \pi_{t,(r)}(E) &= -\sum_{m \neq t} y_m y_t^{-1} \zeta_{t,s} \circ (-1)^{\binom{p}{2}} \Bigg(\frac{{}^\circ\partial_t}{\partial y_{i_1}} \cup \cdots \cup \frac{{}^\circ\partial_t}{\partial y_{i_{j-1}}} \cup \frac{{}^\circ\partial_t}{\partial y_m} \cup \frac{{}^\circ\partial_t}{\partial y_{i_{j+1}}} \\
&\qquad \cup \cdots \cup \frac{{}^\circ\partial_t}{\partial y_{i_p}} \cup {}^\circ\mu_t^{\cup(r-p)} \Bigg) \circ (\zeta_{s,t})^{\otimes(2r-p)} \\
&= -\sum_{m \neq t} y_m y_t^{-1} (-1)^{\binom{p}{2}} \zeta'_{t,s}\left(\frac{{}^\circ\partial_t}{\partial y_{i_1}} \right) \cup \cdots \cup \zeta'_{t,s}\left(\frac{{}^\circ\partial_t}{\partial y_m} \right) \cup \cdots \\
&\qquad \cup \zeta'_{t,s}\left(\frac{{}^\circ\partial_t}{\partial y_{i_p}} \right) \cup \left(\zeta'_{t,s}({}^\circ\mu_t)\right)^{\cup(r-p)} \\
&= -\sum_{m \neq t,s} y_m y_t^{r+(d-1)(r-p)-1} (-1)^{\binom{p}{2}} \frac{{}^\circ\partial_s}{\partial y_{i_1}} \cup \cdots \cup \frac{{}^\circ\partial_s}{\partial y_m} \cup \cdots \cup \frac{{}^\circ\partial_s}{\partial y_{i_p}} \\
&\qquad \cup ({}^\circ\mu_s)^{\cup(r-p)} - y_t^{r+(d-1)(r-p)-2} (-1)^{\binom{p}{2}} \frac{{}^\circ\partial_s}{\partial y_{i_1}} \cup \cdots \\
&\qquad \cup \left(-\sum_{i \neq s} y_t y_i \frac{{}^\circ\partial_s}{\partial y_i} \right) \cup \cdots \cup \frac{{}^\circ\partial_s}{\partial y_{i_p}} \cup ({}^\circ\mu_s)^{\cup(r-p)} \\
&= y_t^{r+(d-1)(r-p)} (-1)^{\binom{p}{2}} \frac{{}^\circ\partial_s}{\partial y_{i_1}} \cup \cdots \cup \frac{{}^\circ\partial_s}{\partial y_t} \cup \cdots \cup \frac{{}^\circ\partial_s}{\partial y_{i_p}} \cup ({}^\circ\mu_s)^{\wedge(r-p)} \\
&= y_t^{r+(d-1)(r-p)} \beta_{s,(r)}(\mathfrak{e}_{i_1\ldots i_p}\mathfrak{s}^{(r-p)}) \\
&= \beta_{s,(r)} \circ \pi_{s,(r)} \circ \xi^{\mathrm{Tot}}_{t,s,(r)}(E).
\end{aligned}$$

If $t \notin \{i_1, \ldots, i_p\}$ and $s = i_l$ for some l, then

$$\begin{aligned}
\beta'_{t,(r)} \circ \pi_{t,(r)}(E) &= (-1)^{\binom{p}{2}} \zeta'_{t,s}\left(\frac{{}^\circ\partial_t}{\partial y_{i_1}} \right) \cup \cdots \cup \zeta'_{t,s}\left(\frac{{}^\circ\partial_t}{\partial y_{i_p}} \right) \cup \left(\zeta'_{t,s}({}^\circ\mu_t)\right)^{\cup(r-p)} \\
&= y_t^{r+(d-1)(r-p)-1} (-1)^{\binom{p}{2}} \frac{{}^\circ\partial_s}{\partial y_{i_1}} \cup \cdots \cup \left(-\sum_{m \neq s} y_t y_m \frac{{}^\circ\partial_s}{\partial y_m} \right) \cup \cdots \\
&\qquad \cup \frac{{}^\circ\partial_s}{\partial y_{i_p}} \cup {}^\circ\mu_s^{\wedge(r-p)} \\
&= -y_t^{r+(d-1)(r-p)} \sum_{m \neq s} y_m (-1)^{\binom{p}{2}} \frac{{}^\circ\partial_s}{\partial y_{i_1}} \cup \cdots \cup \frac{{}^\circ\partial_s}{\partial y_m} \cup \cdots \cup \frac{{}^\circ\partial_s}{\partial y_{i_p}} \\
&\qquad \cup {}^\circ\mu_s^{\cup(r-p)}
\end{aligned}$$

$$
\begin{aligned}
&= -y_t^{r+(d-1)(r-p)} \sum_{m\neq s} y_m \beta_{s,(r)}(\mathfrak{e}_{i_1\dots m \dots i_p}\mathfrak{s}^{(r-p)}) \\
&= \beta_{s,(r)} \circ \pi_{s,(r)} \circ \xi^{\mathrm{Tot}}_{t,s,(r)}(E).
\end{aligned}
$$

If $t = i_j$ and $s = i_l$ for some j, l then

$$
\begin{aligned}
\beta'_{t,(r)} \circ \pi_{t,(r)}(E) &= -\sum_{m\neq t} y_m y_t^{-1} \zeta_{t,s} \circ (-1)^{\binom{p}{2}} \left(\frac{{}^\circ\partial_t}{\partial y_{i_1}} \cup \dots \cup \frac{{}^\circ\partial_t}{\partial y_{i_{j-1}}} \cup \frac{{}^\circ\partial_t}{\partial y_m} \cup \frac{{}^\circ\partial_t}{\partial y_{i_{j+1}}} \right. \\
&\qquad \left. \cup \dots \cup \frac{{}^\circ\partial_t}{\partial y_s} \cup \dots \cup \frac{{}^\circ\partial_t}{\partial y_{i_p}} \cup {}^\circ\mu_t^{\cup(r-p)} \right) \circ (\zeta_{s,t})^{\otimes(2r-p)} \\
&= -\sum_{m\neq t,s} y_m y_t^{-1} (-1)^{\binom{p}{2}} \zeta'_{t,s}\left(\frac{{}^\circ\partial_t}{\partial y_{i_1}} \right) \cup \dots \cup \zeta'_{t,s}\left(\frac{{}^\circ\partial_t}{\partial y_m} \right) \cup \dots \\
&\qquad \cup \zeta'_{t,s}\left(\frac{{}^\circ\partial_t}{\partial y_s} \right) \cup \dots \cup \zeta'_{t,s}\left(\frac{{}^\circ\partial_t}{\partial y_{i_p}} \right) \cup \left(\zeta'_{t,s}({}^\circ\mu_t) \right)^{\cup(r-p)} \\
&= -\sum_{m\neq t,s} y_m y_t^{r+(d-1)(r-p)-2} (-1)^{\binom{p}{2}} \frac{{}^\circ\partial_s}{\partial y_{i_1}} \cup \dots \cup \frac{{}^\circ\partial_s}{\partial y_m} \cup \dots \\
&\qquad \cup \left(-\sum_{i\neq s} y_t y_i \frac{{}^\circ\partial_s}{\partial y_i} \right) \cup \dots \cup \frac{{}^\circ\partial_s}{\partial y_{i_p}} \cup ({}^\circ\mu_s)^{\cup(r-p)} \\
&= y_t^{r+(d-1)(r-p)-1} \sum_{m\neq t,s} \sum_{i\neq s} y_m y_i (-1)^{\binom{p}{2}} \frac{{}^\circ\partial_s}{\partial y_{i_1}} \cup \dots \cup \frac{{}^\circ\partial_s}{\partial y_m} \cup \dots \\
&\qquad \cup \frac{{}^\circ\partial_s}{\partial y_i} \cup \dots \cup \frac{{}^\circ\partial_s}{\partial y_{i_p}} \cup ({}^\circ\mu_s)^{\cup(r-p)} \\
&= y_t^{r+(d-1)(r-p)-1} \sum_{m\neq t,s} y_m y_t (-1)^{\binom{p}{2}} \frac{{}^\circ\partial_s}{\partial y_{i_1}} \cup \dots \cup \frac{{}^\circ\partial_s}{\partial y_m} \cup \dots \\
&\qquad \cup \frac{{}^\circ\partial_s}{\partial y_t} \cup \dots \cup \frac{{}^\circ\partial_s}{\partial y_{i_p}} \cup ({}^\circ\mu_s)^{\cup(r-p)} \\
&= -y_t^{r+(d-1)(r-p)} \sum_{m\neq s} y_m (-1)^{\binom{p}{2}} \frac{{}^\circ\partial_s}{\partial y_{i_1}} \cup \dots \cup \frac{{}^\circ\partial_s}{\partial y_t} \cup \dots \cup \frac{{}^\circ\partial_s}{\partial y_m} \cup \\
&\qquad \dots \cup \frac{{}^\circ\partial_s}{\partial y_{i_p}} \cup ({}^\circ\mu_s)^{\cup(r-p)} \\
&= -y_t^{r+(d-1)(r-p)} \sum_{m\neq s} y_m \beta_{s,(r)}(\mathfrak{e}_{i_1\dots m \dots i_p}\mathfrak{s}^{(r-p)}) \\
&= \beta_{s,(r)} \circ \pi_{s,(r)} \circ \xi^{\mathrm{Tot}}_{t,s,(r)}(E). \qquad \square
\end{aligned}
$$

Therefore we obtain a commutative diagram

$$
\begin{array}{ccc}
\bigoplus_{r\in\mathbb{N}} \mathrm{Tot}(\tau^r \mathcal{K}^{\bullet,\bullet}(\boldsymbol{u}_t, \boldsymbol{v}_t; \mathcal{A}({}_\diamond\sigma, t))) & \xrightarrow{\beta_t\circ\pi_t} & \bar{C}^\bullet(\mathcal{A}(\sigma_\diamond, t), \mathcal{A}({}_\diamond\sigma, t)) \\
\downarrow \zeta_{t,s} & & \downarrow \zeta'_{t,s} \\
\bigoplus_{r\in\mathbb{N}} \mathrm{Tot}(\tau^r \mathcal{K}^{\bullet,\bullet}(\zeta_{t,s}(\boldsymbol{u}_t), \zeta_{t,s}(\boldsymbol{v}_t); \mathcal{A}({}_\diamond\sigma, s))) & \xrightarrow{\beta'_t\circ\pi_t} & \bar{C}^\bullet(\mathcal{A}(\sigma_\diamond, s), \mathcal{A}({}_\diamond\sigma, s)) \\
\downarrow \xi^{\mathrm{Tot}}_{t,s} & & \| \\
\bigoplus_{r\in\mathbb{N}} \mathrm{Tot}(\tau^r \mathcal{K}^{\bullet,\bullet}(\boldsymbol{u}_s, \boldsymbol{v}_s; \mathcal{A}({}_\diamond\sigma, s))) & \xrightarrow{\beta_s\circ\pi_s} & \bar{C}^\bullet(\mathcal{A}(\sigma_\diamond, s), \mathcal{A}({}_\diamond\sigma, s))
\end{array}
$$

where the vertical morphisms are isomorphisms and the horizontal ones are quasi-isomorphisms. Let $\xi'_{t,s} = \xi^{\mathrm{Tot}}_{t,s} \circ \zeta_{t,s}$. The twisting number $r + (d-1)(r-d)$ of the $(r-p, r)$-entry in Figure 2 coincides with the exponent of y_t in the proof of Lemma 4.3. This is equivalent to say that $\xi'_{t,s}$

is the canonical automorphism of $\mathcal{F}^\bullet({}_\diamond\sigma)$ if we write $\mathcal{A}({}_\diamond\sigma)$ in terms of different generators and relations. Moreover, it is easy to check the coherence conditions

$$\xi'_{s,u}\circ\xi'_{t,s}=\xi'_{t,u},\quad \zeta'_{s,u}\circ\zeta'_{t,s}=\zeta'_{t,u}$$

hold true for any additional $u\in\Phi(\sigma_\diamond)$. This gives rise to well-defined morphisms

$$\gamma_\sigma=\beta\circ\pi\colon \mathcal{F}^\bullet({}_\diamond\sigma)\to\bar{C}^\bullet(\mathcal{A}(\sigma_\diamond),\mathcal{A}({}_\diamond\sigma))$$

for all simplices $\sigma\in\mathcal{N}_\bullet(\mathfrak{V})$ which commute with simplicial differentials. Remember that β and π preserve the Hodge decomposition. These facts are summarized as

Theorem 4.4. *Let $\mathcal{E}^{\bullet,\bullet}=\oplus_{r\in\mathbb{N}}\mathcal{E}_r^{\bullet,\bullet}$. The morphisms $\gamma_\sigma\colon\mathcal{F}^\bullet({}_\diamond\sigma)\to\bar{C}^\bullet(\mathcal{A}(\sigma_\diamond),\mathcal{A}({}_\diamond\sigma))$ for all simplices σ on $\mathfrak{V}$ constitute a morphism $\gamma\colon\mathcal{E}^{\bullet,\bullet}\to\bar{\mathbf{C}}'^{\bullet,\bullet}(\mathcal{A})$ of double complexes that gives rise to a quasi-isomorphism $\mathcal{E}^\bullet\to\bar{\mathbf{C}}'^\bullet_{\mathrm{GS}}(\mathcal{A})$. Moreover, γ preserves the Hodge decomposition.*

4.2. **The cotangent complex of a hypersurface.** In [1, Expose VIII] Berthelot defines $\mathbb{L}_{X/Y}$ as a complex concentrated in two degrees when $X\to Y$ factors as a closed immersion $X\to X'$ followed by a smooth morphism $X'\to Y$. Obviously, when $Y=\operatorname{Spec}k$ and $X=\operatorname{Proj}S$, X' can be chosen to be $\operatorname{Proj}R=\mathbb{P}^n$ and so the factorization $X\xrightarrow{\imath}\mathbb{P}^n\to\operatorname{Spec}k$ satisfies the condition. Let $\mathcal{O}=\mathcal{O}_{\mathbb{P}^n}$, and let $\mathcal{I}\subset\mathcal{O}$ be the sheaf of ideals determined by the closed immersion $X\to\mathbb{P}^n$. By definition, $\mathbb{L}^0_{X/k}=\imath^*\Omega_{\mathbb{P}^n}$, $\mathbb{L}^{-1}_{X/k}=\mathcal{I}/\mathcal{I}^2$, and other $\mathbb{L}^j_{X/k}$ are all zero, the differential $\mathcal{I}/\mathcal{I}^2=\imath^*\mathcal{I}\to\imath^*\Omega_{\mathbb{P}^n}$ is induced by $\mathcal{I}\hookrightarrow\mathcal{O}\xrightarrow{\mathrm{d}}\Omega_{\mathbb{P}^n}$.

Note that the complex

$$0\longrightarrow\mathcal{O}_X(-d)\xrightarrow{\partial_{\boldsymbol{u}}}\mathcal{O}_X(-1)^{n+1}\xrightarrow{\partial_{\boldsymbol{v}}}\mathcal{O}_X\longrightarrow 0$$

concentrated in degrees -1, 0 and 1 is the same as $\mathcal{F}_1^{\bullet\vee}[-1]$ where $(-)^\vee=\mathcal{H}om(-,\mathcal{O}_X)$. We claim that the complex presents $\mathbb{L}_{X/k}$. In fact, the isomorphism $\mathcal{I}/\mathcal{I}^2\cong\mathcal{O}_X(-d)$ is obvious. Let $\imath^*$ act on the exact sequence

$$0\longrightarrow\Omega_{\mathbb{P}^n}\longrightarrow\mathcal{O}(-1)^{n+1}\xrightarrow{\partial_{\boldsymbol{v}}}\mathcal{O}\longrightarrow 0. \tag{4.4}$$

Since vector bundles are acyclic for any pullback, we have another exact sequence

$$0\longrightarrow\imath^*\Omega_{\mathbb{P}^n}\longrightarrow\mathcal{O}_X(-1)^{n+1}\xrightarrow{\partial_{\boldsymbol{v}}}\mathcal{O}_X\longrightarrow 0.$$

This immediately gives the quasi-isomorphism

$$\begin{array}{ccccccccccccc}\cdots&\longrightarrow&0&\longrightarrow&\mathcal{O}_X(-d)&\xrightarrow{\partial_{\boldsymbol{u}}}&\mathcal{O}_X(-1)^{n+1}&\xrightarrow{\partial_{\boldsymbol{v}}}&\mathcal{O}_X&\longrightarrow&0&\longrightarrow&\cdots\\ &&\uparrow&&\cong\uparrow&&\uparrow&&\uparrow&&\uparrow&&\\ \cdots&\longrightarrow&0&\longrightarrow&\mathcal{I}/\mathcal{I}^2&\xrightarrow{\mathrm{d}}&\imath^*\Omega_{\mathbb{P}^n}&\longrightarrow&0&\longrightarrow&0&\longrightarrow&\cdots\end{array}$$

Proposition 4.5. *In the derived category $\mathbf{D}(X)$, $\wedge^r\mathbb{L}_{X/k}\cong\mathcal{F}_r^{\bullet\vee}[-r]$ for any $r\in\mathbb{N}$.*

Proof. Because $\mathbb{L}_{X/k}$ is the two-term complex of vector bundles $\mathcal{O}_X(d)\to\imath^*\Omega_{\mathbb{P}^n}$, the exterior power $\wedge^r\mathbb{L}_{X/k}$ is given by

$$\mathcal{O}_X(-dr)\longrightarrow\imath^*\Omega_{\mathbb{P}^n}(-d(r-1))\longrightarrow\cdots\longrightarrow\wedge^{r-s}\imath^*\Omega_{\mathbb{P}^n}(-ds)\longrightarrow\cdots\longrightarrow\wedge^r\imath^*\Omega_{\mathbb{P}^n},$$

for example by [20, §4].

Recall that the exact sequence (4.4) can be generalized to the long exact sequence

$$0\longrightarrow\Omega^l_{\mathbb{P}^n}\longrightarrow\mathcal{O}(-l)^{\binom{n+1}{l}}\xrightarrow{\partial_{\boldsymbol{v}}}\mathcal{O}(-l+1)^{\binom{n+1}{l-1}}\xrightarrow{\partial_{\boldsymbol{v}}}\cdots\xrightarrow{\partial_{\boldsymbol{v}}}\mathcal{O}(-1)^{n+1}\xrightarrow{\partial_{\boldsymbol{v}}}\mathcal{O}\longrightarrow 0$$

for any $l \in \mathbb{N}$.[2] Just as before, the pullback

$$0 \longrightarrow \imath^*\Omega^l_{\mathbb{P}^n} \longrightarrow \mathcal{O}_X(-l)^{\binom{n+1}{l}} \xrightarrow{\partial_{\boldsymbol{v}}} \mathcal{O}_X(-l+1)^{\binom{n+1}{l-1}} \xrightarrow{\partial_{\boldsymbol{v}}} \cdots \xrightarrow{\partial_{\boldsymbol{v}}} \mathcal{O}_X(-1)^{n+1} \xrightarrow{\partial_{\boldsymbol{v}}} \mathcal{O}_X \longrightarrow 0$$

is also exact.

These complexes constitute the diagram as follows,

$$\begin{array}{ccccccccc}
 & & & & & & & & \mathcal{O}_X \\
 & & & & & & & & \uparrow{\scriptstyle \partial_{\boldsymbol{v}}} \\
 & & & & & & \mathcal{O}_X(-d) & \xrightarrow{\partial_{\boldsymbol{u}}} & \mathcal{O}_X(-1)^{n+1} \\
 & & & & & & \uparrow{\scriptstyle \partial_{\boldsymbol{v}}} & & \uparrow{\scriptstyle \partial_{\boldsymbol{v}}} \\
 & & & & \iddots & & \vdots & & \vdots \\
 & & & & & & \uparrow{\scriptstyle \partial_{\boldsymbol{v}}} & & \uparrow{\scriptstyle \partial_{\boldsymbol{v}}} \\
 & & \mathcal{O}_X(d-rd) & \xrightarrow{\partial_{\boldsymbol{u}}} & \cdots & \xrightarrow{\partial_{\boldsymbol{u}}} & \mathcal{O}_X(2-r-d)^{\binom{n+1}{r-2}} & \xrightarrow{\partial_{\boldsymbol{u}}} & \mathcal{O}_X(-r+1)^{\binom{n+1}{r-1}} \\
 & & \uparrow{\scriptstyle \partial_{\boldsymbol{v}}} & & & & \uparrow{\scriptstyle \partial_{\boldsymbol{v}}} & & \uparrow{\scriptstyle \partial_{\boldsymbol{v}}} \\
\mathcal{O}_X(-rd) & \xrightarrow{\partial_{\boldsymbol{u}}} & \mathcal{O}_X(d-rd-1)^{n+1} & \xrightarrow{\partial_{\boldsymbol{u}}} & \cdots & \xrightarrow{\partial_{\boldsymbol{u}}} & \mathcal{O}_X(1-r-d)^{\binom{n+1}{r-1}} & \xrightarrow{\partial_{\boldsymbol{u}}} & \mathcal{O}_X(-r)^{\binom{n+1}{r}} \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
\mathcal{O}_X(-rd) & \longrightarrow & \imath^*\Omega_{\mathbb{P}^n}(d-rd) & \longrightarrow & \cdots & \longrightarrow & \imath^*\Omega^{r-1}_{\mathbb{P}^n}(-d) & \longrightarrow & \imath^*\Omega^r_{\mathbb{P}^n}
\end{array}$$

where each column is exact, and each square is anti-commutative (we adapt the Koszul sign rule here). Note that the associated total complex of the double complex by deleting the bottom row is exactly $\mathcal{F}^{\bullet\vee}_r[-r]$. Hence the diagram gives rise to a quasi-isomorphisms $\wedge^r\mathbb{L}_{X/k} \to \mathcal{F}^{\bullet\vee}_r[-r]$. $\square$

Before closing this section, let us compare Buchweitz and Flenner's formula (4.1)and ours (i.e. $HH^i(X) \cong H^i(\mathcal{H}^\bullet)$) via the isomorphisms $\wedge^r\mathbb{L}_{X/k} \to \mathcal{F}^{\bullet\vee}_r[-r]$. Since $\mathcal{F}^\bullet_q$ is a complex of locally free sheaves, one easily deduces that $\operatorname{Ext}^p_X(\mathcal{F}^{\bullet\vee}_q[-q], \mathcal{O}_X) \cong \mathbb{H}^{p+q}(\mathcal{F}^\bullet_q)$ where the hypercohomology $\mathbb{H}^{p+q}(\mathcal{F}^\bullet_q)$ can also be computed by the (total) Čech complex (see e.g. [3, Ch. 1]), namely, $\mathbb{H}^{p+q}(\mathcal{F}^\bullet_q) \cong H^{p+q}(\mathcal{H}^\bullet_q)$. So

$$\bigoplus_{p+q=i} \operatorname{Ext}^p_X(\wedge^q\mathbb{L}_{X/k}, \mathcal{O}_X) \cong \bigoplus_{p+q=i} H^{p+q}(\mathcal{H}^\bullet_q) = H^i(\mathcal{H}^\bullet).$$

Thus the Hodge decomposition and the HKR decomposition (in the sense of Buchweitz and Flenner) of $HH^\bullet(X)$ are component-wise isomorphic for any hypersurface $X \subset \mathbb{P}^n$.

5. Cohomology computation

In this section we prove our main theorem (Theorem 1.1), providing a computation of the Hochschild cohomology groups of a projective hypersurface of degree d in $\mathbb{P}^n$ in terms of the easier complexes $\mathcal{H}^\bullet_r$. The result makes a basic distinction between the case $d > n+1$, the harder case $d = n+1$ and the easier case $d \leq n$.

Let us associate some graded modules to $X = \operatorname{Proj} S$. Note that the $\partial_{\boldsymbol{v}}$ constitute a morphism

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathcal{K}^{-3}(\boldsymbol{u}; S) & \xrightarrow{\partial_{\boldsymbol{u}}} & \mathcal{K}^{-2}(\boldsymbol{u}; S) & \xrightarrow{\partial_{\boldsymbol{u}}} & \mathcal{K}^{-1}(\boldsymbol{u}; S) & \xrightarrow{\partial_{\boldsymbol{u}}} & \mathcal{K}^{0}(\boldsymbol{u}; S) \\
 & & \uparrow{\scriptstyle \partial_{\boldsymbol{v}}} & & \uparrow{\scriptstyle \partial_{\boldsymbol{v}}} & & \uparrow{\scriptstyle \partial_{\boldsymbol{v}}} & & \uparrow \\
\cdots & \longrightarrow & \mathcal{K}^{-2}(\boldsymbol{u}; S) & \xrightarrow{\partial_{\boldsymbol{u}}} & \mathcal{K}^{-1}(\boldsymbol{u}; S) & \xrightarrow{\partial_{\boldsymbol{u}}} & \mathcal{K}^{0}(\boldsymbol{u}; S) & \longrightarrow & 0
\end{array}$$

[2]The proof is similar to the one of [11, Thm. 8.13].

from which we obtain the cokernel complex $\mathcal{C}^\bullet(\boldsymbol{u};S)$:

$$\cdots \longrightarrow \mathcal{K}^{-3}(\boldsymbol{u};S)/\operatorname{im}\partial_{\boldsymbol{v}} \xrightarrow{\partial_{\boldsymbol{u}}} \mathcal{K}^{-2}(\boldsymbol{u};S)/\operatorname{im}\partial_{\boldsymbol{v}} \xrightarrow{\partial_{\boldsymbol{u}}} \mathcal{K}^{-1}(\boldsymbol{u};S)/\operatorname{im}\partial_{\boldsymbol{v}} \xrightarrow{\partial_{\boldsymbol{u}}} \mathcal{K}^{0}(\boldsymbol{u};S). \tag{5.1}$$

The i-th cohomology group of $\mathcal{C}^\bullet(\boldsymbol{u};S)$ is denoted by P^i and the i-th cocycle group by Q^i. Clearly, the S-modules P^i, Q^i are graded modules. Denote by Z^i the i-th cocycle group of $\mathcal{K}^\bullet(\boldsymbol{v};R)$, which is a graded R-module.

Observe that we have defined quasi-isomorphisms $\mathcal{H}^\bullet \to \mathcal{G}^\bullet \to \mathcal{E}^\bullet \to \bar{\mathbf{C}}'^\bullet_{\mathrm{GS}}(\mathcal{A})$. From now on, let us compute $H^\bullet_{\mathrm{GS}}(\mathcal{A}) := H^\bullet\bar{\mathbf{C}}'^\bullet_{\mathrm{GS}}(\mathcal{A})$ by using $\mathcal{H}^{\bullet,\bullet}$. We need some lemmas.

Lemma 5.1. *The cohomology groups of $\mathcal{K}^\bullet(\boldsymbol{v}^\star;S)$ are $H^0 = H^0_0 = k$, $H^{-1} = H^{-1}_{d-1} = k\boldsymbol{u}^\star$ where $\boldsymbol{u}^\star = (\partial F/\partial x_0, -\partial F/\partial x_1, \dots, (-1)^n\partial F/\partial x_n)$, and $H^i = 0$ for all $i \neq 0, -1$.*

Proof. Recall Remark 2.1 and note that $\boldsymbol{v}^\star_0 = (-x_1, x_2, \dots, (-1)^n x_n)$ is a regular sequence in S. So $\mathcal{K}^\bullet(\boldsymbol{v}^\star;S)$, which is the mapping cone of $\mathcal{K}^\bullet(\boldsymbol{v}^\star_0;S) \xrightarrow{x_0} \mathcal{K}^\bullet(\boldsymbol{v}^\star_0;S)$, is quasi-isomorphic to

$$\cdots \longrightarrow 0 \longrightarrow S/(\boldsymbol{v}^\star_0) \xrightarrow{x_0} S/(\boldsymbol{v}^\star_0) \longrightarrow 0.$$

Since $S/(\boldsymbol{v}^\star_0) \cong k[x_0]/(x_0^d)$, we have $H^0 = k$ and $H^{-1} \cong k(1-d)$ as graded modules. To show $\boldsymbol{u}^\star$ is a base element in H^{-1}, we will check that $\boldsymbol{u}^\star$ never belongs to $\operatorname{im}\partial_{\boldsymbol{v}^\star}$. This is clear since $\partial F/\partial x_0$ contains dx_0^{d-1} as a summand. □

The following is well known:

Lemma 5.2. *The cohomology groups of $\mathcal{K}^\bullet(\boldsymbol{v};R)$ are $H^0 = H^0_0 = k$, and $H^i = 0$ for all $i \neq 0$.*

Lemma 5.3. *Let τ^r_0 be the zeroth graded component of $\tau^r\mathcal{K}^{\bullet,\bullet}(\boldsymbol{u},\boldsymbol{v};S)$.*

(1) *If $0 \le r \le n$, then*

$$H^i(\operatorname{Tot}\tau^r_0) \cong \begin{cases} 0, & 0 \le i < r, \\ Q^{-r}_r, & i = r, \\ P^{i-2r}_{r+(i-r)(d-1)}, & r < i \le 2r. \end{cases}$$

(2) *If $r \ge n+1$ and $d = n+1$, then*

$$H^i(\operatorname{Tot}\tau^r_0) \cong \begin{cases} 0, & 0 \le i \le r,\ i \neq n, \\ k, & i = n, \\ P^{i-2r}_{r+(i-r)(d-1)}, & r < i \le 2r. \end{cases}$$

(3) *If $r \ge n+1$ and $d \neq n+1$, then*

$$H^i(\operatorname{Tot}\tau^r_0) \cong \begin{cases} 0, & 0 \le i \le r, \\ P^{i-2r}_{r+(i-r)(d-1)}, & r < i \le 2r. \end{cases}$$

Proof. We prove the statements by computing the spectral sequence ${}^I E^{p,q}_a$ determined by τ^r_0.

(1) Let $0 \le r \le n$. The p-th column of $\tau^r\mathcal{K}^{\bullet,\bullet}(\boldsymbol{u},\boldsymbol{v};S)$ is the truncation $\tau^{\le -(n+1-r+p)}\mathcal{K}^\bullet(\boldsymbol{v}^\star;S)$ up to twist. Notice that $-(n+1-r+p) \le -1$. By Lemma 5.1, $H^i(\tau^{\le -(n+1-r+p)}\mathcal{K}^\bullet(\boldsymbol{v}^\star;S)) = 0$ if $i \neq -(n+1-r+p)$. It follows that the p-th column of $\tau^r\mathcal{K}^{\bullet,\bullet}(\boldsymbol{u},\boldsymbol{v};S)$ is exact except in spot (p,r). By considering the zeroth graded component, we have ${}^I E^{p,q}_1 = 0$ if $q \neq r$, and

$${}^I E^{p,r}_1 = \big(S(r+p(d-1))^{\binom{n+1}{r-p}}/\operatorname{im}\partial_{\boldsymbol{v}}\big)_0 = \big(\mathcal{K}^{-(r-p)}(\boldsymbol{u};S)/\operatorname{im}\partial_{\boldsymbol{v}}\big)_{r+p(d-1)}.$$

To compute ${}^I E^{p,r}_2$, it suffices to consider the complex

$$\big(\mathcal{K}^{-r}(\boldsymbol{u};S)/\operatorname{im}\partial_{\boldsymbol{v}}\big)_r \longrightarrow \cdots \longrightarrow \big(\mathcal{K}^{-1}(\boldsymbol{u};S)/\operatorname{im}\partial_{\boldsymbol{v}}\big)_{r+(r-1)(d-1)} \xrightarrow{\partial_{\boldsymbol{u}}} \big(\mathcal{K}^{0}(\boldsymbol{u};S)\big)_{rd}.$$

Comparing this complex with (5.1), we have ${}^{I}E_2^{0,r} = Q_r^{-r}$, ${}^{I}E_2^{p,r} = P_{r+p(d-1)}^{-(r-p)}$ if $p \geq 1$. Hence $H^i(\operatorname{Tot}\tau_0^r) \cong {}^{I}E_\infty^{i-r,r} = {}^{I}E_2^{i-r,r} = P_{r+(i-r)(d-1)}^{i-2r}$ when $r < i \leq 2r$, and $H^r(\operatorname{Tot}\tau_0^r) \cong {}^{I}E_\infty^{0,r} = {}^{I}E_2^{0,r} = Q_r^{-r}$.

(2) Let $r \geq n+1$ and $d = n+1$. Just like in the situation in (1), we have

$$ {}^{I}E_1^{p,r} = \big(\mathcal{K}^{-(r-p)}(\boldsymbol{u}; S)/\operatorname{im}\partial_{\boldsymbol{v}}\big)_{r+p(d-1)}. $$

By Lemma 5.1 and taking into consideration the degrees, we find one more nonzero ${}^{I}E_1^{p,q}$, namely, ${}^{I}E_1^{0,n} \cong k$. For ${}^{I}E_2^{p,q}$, as shown in (1), ${}^{I}E_2^{0,r} = Q_r^{-r}$ and ${}^{I}E_2^{p,r} = P_{r+p(d-1)}^{-(r-p)}$ for all $1 \leq p \leq 2r$. Note that $Q_{n+1}^{-(n+1)}$ is a k-submodule of $(S/\operatorname{im}\partial_{\boldsymbol{v}})_{n+1} = k_{n+1} = 0$ and $Q^{-s} = 0$ if $s \geq n+2$. Hence ${}^{I}E_2^{0,r} = Q_r^{-r} = 0$ since $r \geq n+1$. We also have ${}^{I}E_2^{0,n} = {}^{I}E_1^{0,n} \cong k$ and the rest ${}^{I}E_2^{p,q}$ being all zero. Assertion (2) follows.

(3) The proof is completely similar to (2). The only difference is that ${}^{I}E_1^{0,n}$ is zero since $d \neq n+1$. □

The double complex $\mathcal{H}_r^{\bullet,\bullet}$ leads to a spectral sequence ${}^{II}E_{r,a}^{p,q}$ by filtration by rows. We begin to calculate it.

5.1. **Case 1:** $d > n+1$. Suppose $m \geq 0$ and $\mathcal{O} = \mathcal{O}_{\mathbb{P}^n}$. By the exact sequence

$$ 0 \longrightarrow \mathcal{O}(m-d) \xrightarrow{F} \mathcal{O}(m) \longrightarrow \mathcal{O}_X(m) \longrightarrow 0, $$

we immediately conclude that $H^i(X, \mathcal{O}_X(m)) = 0$ if $i \neq 0,\ n-1$, and $H^{n-1}(X, \mathcal{O}_X(m)) \cong H^n(\mathbb{P}^n, \mathcal{O}(m-d)) \cong H^0(\mathbb{P}^n, \mathcal{O}(d-n-1-m))^*$. Obviously, $H^0(\mathbb{P}^n, \mathcal{O}(d-n-1-m))$ has a basis

$$ \{x_0^{i_0}x_1^{i_1}\cdots x_n^{i_n} \in R \mid i_0 + i_1 + \cdots + i_n = d-n-1-m,\ i_0, i_1, \ldots, i_n \geq 0\}. $$

On the other hand, the Čech cohomology group $\check{H}^{n-1}(\mathfrak{U}, \mathcal{O}_X(m))$ has a basis

$$ \{x_0^{j_0}x_1^{j_1}\cdots x_n^{j_n} \in S_{x_1\cdots x_n} \mid j_0 + j_1 + \cdots + j_n = m,\ 0 \leq j_0 \leq d-1,\ j_1, \ldots, j_n \leq -1\} $$

where $S_{x_1\cdots x_n}$ is the localization of S at $x_1 \cdots x_n$. Since both groups have finite dimension over k, the duality gives rise to the bijection

$$ \begin{aligned} \mathscr{S}\colon H^0(\mathbb{P}^n, \mathcal{O}(d-n-1-m)) &\longrightarrow \check{H}^{n-1}(\mathfrak{U}, \mathcal{O}_X(m)), \\ x_0^{i_0}x_1^{i_1}\cdots x_n^{i_n} &\longmapsto x_0^{d-1-i_0}x_1^{-1-i_1}\cdots x_n^{-1-i_n}. \end{aligned} \tag{5.2} $$

The map $\mathscr{S}$ induces $H^0(\mathbb{P}^n, \mathcal{O}(d-n-1-m)^r) \longrightarrow \check{H}^{n-1}(\mathfrak{U}, \mathcal{O}_X(m)^r)$ for any $r \in \mathbb{N}$ which is also denoted by $\mathscr{S}$.

Since $\check{H}^{n-1}(\mathfrak{U}, \mathcal{O}_X(m)) = 0$ if $m \geq d$, by the definition of $\mathcal{H}_r^{\bullet,\bullet}$, we have

$$ {}^{II}E_{r,1}^{p,q} = \check{H}^q(\mathfrak{U}, \mathcal{F}_r^p) = \begin{cases} \check{H}^{n-1}\big(\mathfrak{U}, \mathcal{O}_X(p)^{\binom{n+1}{p}}\big), & 0 \leq p \leq r,\ q = n-1, \\ (\operatorname{Tot}\tau_0^r)^p, & 0 \leq p \leq 2r,\ q = 0, \\ 0, & \text{otherwise.} \end{cases} $$

Since $\check{H}^{n-1}\big(\mathfrak{U}, \mathcal{O}_X(p)^{\binom{n+1}{p}}\big) = \big(R_{d-n-1-p}^{\binom{n+1}{p}}\big)^* = \mathcal{K}^{-p}(\boldsymbol{v}; R)_{d-n-1-p}^*$, the complex

$$ {}^{II}E_{r,1}^{0,n-1} \longrightarrow {}^{II}E_{r,1}^{1,n-1} \longrightarrow \cdots \longrightarrow {}^{II}E_{r,1}^{r-1,n-1} \longrightarrow {}^{II}E_{r,1}^{r,n-1} $$

is dual to

$$ \mathcal{K}^0(\boldsymbol{v}; R)_{d-n-1} \longleftarrow \mathcal{K}^{-1}(\boldsymbol{v}; R)_{d-n-2} \longleftarrow \cdots \longleftarrow \mathcal{K}^{-r+1}(\boldsymbol{v}; R)_{d-n-r} \longleftarrow \mathcal{K}^{-r}(\boldsymbol{v}; R)_{d-n-1-r}. \tag{5.3} $$

By Lemma 5.2, the only non trivial cohomology of the complex $\mathcal{K}^\bullet(\boldsymbol{v}; R)$ is $H^0(\mathcal{K}^\bullet(\boldsymbol{v}; R)) = k$. The zero-th cohomology group of (5.3) is zero since the $(d-n-1)$-st graded component in k is zero. The unique possible nonzero cohomology of (5.3) is $H^{-r} = Z^{-r}_{d-n-1-r}$, yielding ${}^{II}E^{r,n-1}_{r,2} = \mathscr{S}(Z^{-r}_{d-n-1-r})$. Combining this with Lemma 5.3, we obtain that ${}^{II}E^{p,q}_{r,2}$ is given by

$$ {}^{II}E^{p,q}_{r,2} = \begin{cases} \mathscr{S}(Z^{-r}_{d-n-1-r}), & p=r,\ q=n-1, \\ P^{p-2r}_{r+(p-r)(d-1)}, & r<p\le 2r,\ q=0, \\ Q^{-r}_r, & p=r,\ q=0, \\ 0, & \text{otherwise.} \end{cases} $$

Immediately, ${}^{II}E^{p,q}_{r,n} = \cdots = {}^{II}E^{p,q}_{r,2}$. On one hand, ${}^{II}E^{r+n,0}_{r,n} = 0$ when $r \le n-1$, since $r+n > 2r$; on the other hand, in case $r \ge n$, we have $d-n-1-r \le -1$ and so ${}^{II}E^{r,n-1}_{r,n} = 0$ since R has only non-negative grading. So in order to show ${}^{II}E^{p,q}_{r,n+1} = {}^{II}E^{p,q}_{r,n}$ for any pair (r,n), it is sufficient to prove the differential ${}^{II}E^{n,n-1}_{n,n} \to {}^{II}E^{2n,0}_{n,n}$ (i.e. the case $r=n$) is zero.

Since ${}^{II}E^{n,n-1}_{n,n}$ is a sub-quotient of $\check{\mathbf{C}}'^{n-1}(\mathfrak{U}, \mathcal{F}^n_n)$, we choose a cocycle $c^{n-1,n} \in \check{\mathbf{C}}'^{n-1}(\mathfrak{U}, \mathcal{F}^n_n)$ for any class in ${}^{II}E^{n,n-1}_{n,n}$. Performing a diagram chase, a cochain $(c^{0,2n-1}, c^{1,2n-2}, \dots, c^{n-1,n})$ in $\mathcal{H}^\bullet$ can be given. Notice that $c^{0,2n-1} \in \mathcal{H}^{0,2n-1} = \check{\mathbf{C}}'^0(\mathfrak{U}, \mathcal{F}^{2n-1}_n) = \mathcal{K}^{-1}(\boldsymbol{u}; S)_{nd-d+1}$, and so $d_{v,\mathcal{H}}(c^{0,2n-1}) = \partial_{\boldsymbol{u}}(c^{0,2n-1})$ is a coboudary in $\mathcal{K}^0(\boldsymbol{u}; S)_{nd}$, i.e. $d_{v,\mathcal{H}}(c^{0,2n-1})$ represents the zero class in $P^0_{nd} = {}^{II}E^{2n,0}_{n,n}$. It follows that the differential ${}^{II}E^{n,n-1}_{n,n} \to {}^{II}E^{2n,0}_{n,n}$ is a zero map. Therefore, ${}^{II}E^{p,q}_{r,\infty} = {}^{II}E^{p,q}_{r,2}$, and

$$ H^i(\mathcal{H}^\bullet) \cong \bigoplus_{r\in\mathbb{N}} \bigoplus_{p+q=i} {}^{II}E^{p,q}_{r,\infty} = \bigoplus_{r<i} P^{i-2r}_{r+(i-r)(d-1)} \oplus Q^{-i}_i \oplus \mathscr{S}(Z^{-i+n-1}_{d-i-2}). $$

5.2. **Case 2:** $d = n+1$. The formula

$$ {}^{II}E^{p,q}_{r,1} = \begin{cases} \check{H}^{n-1}\big(\mathfrak{U}, \mathcal{O}_X(p)^{\binom{n+1}{p}}\big), & 0\le p\le r,\ q=n-1, \\ (\operatorname{Tot}\tau^r_0)^p, & 0\le p\le 2r,\ q=0, \\ 0, & \text{otherwise.} \end{cases} $$

remains valid in this case. Note that the complex (5.3) has only one nonzero term $\mathcal{K}^0(\boldsymbol{v}; R)_{d-n-1} = R_0 = k$. By applying Lemma 5.3 again, we conclude that for $0 \le r \le n$,

$$ {}^{II}E^{p,q}_{r,2} = \begin{cases} k, & p=0,\ q=n-1, \\ P^{p-2r}_{r+n(p-r)}, & r<p\le 2r,\ q=0, \\ Q^{-r}_r, & p=r,\ q=0, \\ 0, & \text{otherwise,} \end{cases} $$

and for $r \ge n+1$,

$$ {}^{II}E^{p,q}_{r,2} = \begin{cases} k, & p=0,\ q=n-1, \\ k, & p=n,\ q=0, \\ P^{p-2r}_{r+n(p-r)}, & r<p\le 2r,\ q=0, \\ 0. & \text{otherwise.} \end{cases} $$

It follows that ${}^{II}E^{p,q}_{r,n} = \cdots = {}^{II}E^{p,q}_{r,2}$.

Since for any $V_{i_1\dots i_s} \in \mathfrak{V}$, the algebra $A_{i_1\dots i_s} = \mathcal{O}_X(V_{i_1\dots i_s})$ is identified with the zero-th graded component of $S_{x_{i_1}\cdots x_{i_s}}$, the localization of S with respect to the element $x_{i_1}\cdots x_{i_s}$, we conclude that the Čech complex $\check{\mathbf{C}}'^\bullet(\mathfrak{U}, \mathcal{F}^0_r)$ for any r is the sub-complex of

$$ \prod_{i_1} S_{x_{i_1}} \longrightarrow \prod_{i_1<i_2} S_{x_{i_1}x_{i_2}} \longrightarrow \cdots \longrightarrow \prod_{i_1<\cdots<i_{n-1}} S_{x_{i_1}\cdots x_{i_{n-1}}} \longrightarrow S_{x_1\cdots x_n} $$

consisting of all cochains of degree zero. Since ${}^{II}E^{0,n-1}_{r,n}$ is a sub-quotient of $\check{\mathbf{C}}'^{n-1}(\mathfrak{U}, \mathcal{F}^0_r)$, it seems apt to choose $x_0^n x_1^{-1}\cdots x_n^{-1} \in \check{\mathbf{C}}'^{n-1}(\mathfrak{U}, \mathcal{F}^0_r)$ as a base element of ${}^{II}E^{0,n-1}_{r,n}$. However, for the sake of easy computation, we use $x_1^{-1}\cdots x_n^{-1}\cdot \partial F/\partial x_0$ flexibly rather than $x_0^n x_1^{-1}\cdots x_n^{-1}$. Similar to the argument in the case $d > n+1$, one finds a cochain $(c^{0,n-1}, c^{1,n-2}, \dots, c^{n-1,0})$ in $\mathcal{H}^\bullet$ with $c^{n-1,0} = x_1^{-1}\cdots x_n^{-1}\cdot \partial F/\partial x_0$. The differential ${}^{II}E^{0,n}_{r,n-1} \to {}^{II}E^{n,0}_{r,n}$ sends the class represented by $c^{n-1,0}$ to the one represented by $d_{v,\mathcal{H}}(c^{0,n-1})$.

(1) If $0 \le r \le n-1$, $d_{v,\mathcal{H}}(c^{0,n-1})$ belongs to $P^{n-2r}_{r+n(n-r)}$. Recall the shape and size of the triangle $\tau^r\mathcal{K}^{\bullet,\bullet}(\boldsymbol{u},\boldsymbol{v};S)$. The element $d_{v,\mathcal{H}}(c^{0,n-1})$ is zero itself if r is very small, or is a sum $\partial_{\boldsymbol{u}}(?) + \partial_{\boldsymbol{v}}(?')$ if r is larger. According to the construction of (5.3), $\partial_{\boldsymbol{u}}(?) + \partial_{\boldsymbol{v}}(?')$ necessarily represents the zero class. In both cases, $c^{n-1,0}$ is killed by the differential ${}^{II}E^{0,n}_{r,n-1} \to {}^{II}E^{n,0}_{r,n}$.

(2) If $r = n$, the diagram chase shows $d_{v,\mathcal{H}}(c^{0,n-1}) = \boldsymbol{u}^\star + \operatorname{im}\partial_{\boldsymbol{v}} \in Q^{-n}_n = \ker\{S^{n+1}_n/\operatorname{im}\partial_{\boldsymbol{v}} \to S^{n(n+1)/2}_{2n}\operatorname{im}\partial_{\boldsymbol{v}}\}$. By the definition of $\mathcal{C}^\bullet(\boldsymbol{u};S)$, $\boldsymbol{u}^\star + \operatorname{im}\partial_{\boldsymbol{v}}$ happens to be a base element of $\operatorname{im}\{S_0/\operatorname{im}\partial_{\boldsymbol{v}} \to S^{n+1}_n/\operatorname{im}\partial_{\boldsymbol{v}}\}$. So ${}^{II}E^{0,n}_{r,n-1} = k \to {}^{II}E^{n,0}_{r,n} = Q^{-n}_n$ is injective and its cokernel is given by $Q^{-n}_n/(k\boldsymbol{u}^\star + \operatorname{im}\partial_{\boldsymbol{v}}) = P^{-n}_n$.

(3) If $r \ge n+1$, we claim that the differential ${}^{II}E^{0,n}_{r,n-1} = k \to {}^{II}E^{n,0}_{r,n} = k$ is an isomorphism. The assertion follows from Lemma 5.4 which will be proven later on.

Summarizing, the spectral sequence

$$
\begin{aligned}
{}^{II}E^{p,q}_{r,\infty} = {}^{II}E^{p,q}_{r,n+1} &= \begin{cases} k, & p=0,\ q=n-1,\\ P^{p-2r}_{r+n(p-r)}, & r<p\le 2r,\ q=0,\\ Q^{-r}_r, & p=r,\ q=0,\\ 0, & \text{otherwise,}\end{cases} && \text{if } 0\le r\le n-1,\\
{}^{II}E^{p,q}_{r,\infty} = {}^{II}E^{p,q}_{r,n+1} &= \begin{cases} P^{p-2r}_{r+n(p-r)}, & r\le p\le 2r,\ q=0,\\ 0, & \text{otherwise,}\end{cases} && \text{if } r=n,\\
{}^{II}E^{p,q}_{r,\infty} = {}^{II}E^{p,q}_{r,n+1} &= \begin{cases} P^{p-2r}_{r+n(p-r)}, & r< p\le 2r,\ q=0,\\ 0, & \text{otherwise,}\end{cases} && \text{if } r\ge n+1.
\end{aligned}
$$

Therefore,

$$
H^i(\mathcal{H}^\bullet) \cong \begin{cases} \bigoplus_{r<i} P^{i-2r}_{r+n(i-r)} \oplus Q^{-i}_i, & i \ne n-1, n,\\ \bigoplus_{r<i} P^{i-2r}_{r+n(i-r)} \oplus Q^{-i}_i \oplus k^n, & i = n-1,\\ \bigoplus_{r\le i} P^{i-2r}_{r+n(i-r)}, & i=n.\end{cases}
$$

Note that $\mathcal{F}^q_r$ is a direct sum of some terms as given in Figure 2, and hence $\mathcal{H}^{p,q}_r$ admits a decomposition

$$
\check{\mathbf{C}}^p(\mathfrak{U}, \mathcal{O}_X(q)^{\binom{n+1}{q}}) \oplus \check{\mathbf{C}}^p(\mathfrak{U}, \mathcal{O}_X(q+d-2)^{\binom{n+1}{q-2}}) \oplus \check{\mathbf{C}}^p(\mathfrak{U}, \mathcal{O}_X(q+2d-4)^{\binom{n+1}{q-4}}) \oplus \cdots
$$

when $q \le r$. Intuitively, $\mathcal{O}_X(q)^{\binom{n+1}{q}}$ appearing in the first component corresponds to a graded module located at the leftmost edge in Figure 2. We hence call a cochain in $\mathcal{H}^{p,q}_r$ *left preferred* if it has possible nonzero component only in $\check{\mathbf{C}}^p(\mathfrak{U}, \mathcal{O}_X(q)^{\binom{n+1}{q}})$.

Lemma 5.4. *Suppose $d = n+1$ and $r \ge n$. There exists a cochain $(c^{0,n-1}, c^{1,n-2}, \ldots, c^{n-1,0})$ in $\mathcal{H}^{n-1}_r$ such that each $c^{n-1-q,q}$ is left preferred in $\mathcal{H}^{n-1-q,q}_r$ and*

$$
c^{n-1,0} = x_1^{-1}\cdots x_n^{-1}\frac{\partial F}{\partial x_0}, \quad d_{\mathcal{F}}(c^{0,n-1}, c^{1,n-2}, \ldots, c^{n-1,0}) = ((-1)^{n-1}\boldsymbol{u}^\star, 0, \ldots, 0).
$$

Proof. During the proof, we will frequently meet elements in $S_{x_{i_1}\cdots x_{i_m}}$. To avoid confusion, we underline denominators to distinguish between similar looking elements. For example, $\underline{x_1^{-1}} \in S_{x_1}$, $\underline{x_1^{-1}x_2^0} \in S_{x_1x_2}$, $\underline{x_1^{-1}x_2^0x_3^0} \in S_{x_1x_2x_3}$. The notations $\mathfrak{f}_{j_1\ldots j_s}$ stand for formal bases elements.

When the Čech indices $(i_1, \ldots, i_s)$ appear, the complements are denote by $(j_1, \ldots, j_{n-s})$, namely, the latter are obtained by deleting $i_1, \ldots, i_s$ from $(1, 2, \ldots, n)$. The permutation

$$
\begin{pmatrix} 1 & \ldots & s & s+1 & \ldots & n\\ i_1 & \ldots & i_s & j_1 & \ldots & j_{n-s}\end{pmatrix}
$$

is a shuffle, whose parity $(n^2-s^2+n-s)/2-(j_1+\cdots+j_{n-s})$ is denoted by $\wp(i_1,\ldots,i_s)$ or even by $\wp(\imath)$ if no confusion arises.

Starting with $c^{n-1,0}=\underline{x_1^{-1}\cdots x_n^{-1}}\partial F/\partial x_0$, we have

$$\begin{aligned}d_{\mathcal{F}}(c^{n-1,0})&=(-1)^{n-1}\underline{x_1^{-1}\cdots x_n^{-1}}x_0\frac{\partial F}{\partial x_0}\mathfrak{f}_0+(-1)^{n-1}\sum_{j=1}^{n}\underline{x_1^{-1}\cdots x_j^{0}\cdots x_n^{-1}}\frac{\partial F}{\partial x_0}\mathfrak{f}_j\\&=(-1)^{n-1}\sum_{j=1}^{n}\underline{x_1^{-1}\cdots x_j^{0}\cdots x_n^{-1}}\left(\frac{\partial F}{\partial x_0}\mathfrak{f}_j-\frac{\partial F}{\partial x_j}\mathfrak{f}_0\right).\end{aligned}$$

Choose $c^{n-2,1}=(c^{n-2,1}_{i_1,\ldots,i_{n-1}})$ as

$$c^{n-2,1}_{i_1,\ldots,i_{n-1}}=(-1)^{\wp(\imath)+1}\underline{x_{i_1}^{-1}\cdots x_{i_{n-1}}^{-1}}\left(\frac{\partial F}{\partial x_0}\mathfrak{f}_{j_1}-\frac{\partial F}{\partial x_{j_1}}\mathfrak{f}_0\right).$$

One can easily show that $\partial_{\boldsymbol{u}}(c^{n-2,1})=0$. Thus $d_{\mathcal{F}}(c^{n-2,1})=(-1)^{n-2}\partial_{\boldsymbol{v}}(c^{n-2,1})$ whose components are

$$\begin{aligned}d_{\mathcal{F}}(c^{n-2,1}_{i_1,\ldots,i_{n-1}})&=(-1)^{j_1+1}\underline{x_{i_1}^{-1}\cdots x_{i_{n-1}}^{-1}}x_0\frac{\partial F}{\partial x_0}\mathfrak{f}_{0j_1}+(-1)^{j_1+1}\sum_{l=1}^{n-1}\underline{x_{i_1}^{-1}\cdots x_{i_l}^{0}\cdots x_{i_{n-1}}^{-1}}\frac{\partial F}{\partial x_0}\mathfrak{f}_{i_lj_1}\\&\quad+(-1)^{j_1+1}\sum_{l=1}^{n-1}\underline{x_{i_1}^{-1}\cdots x_{i_l}^{0}\cdots x_{i_{n-1}}^{-1}}\frac{\partial F}{\partial x_{j_1}}\mathfrak{f}_{0i_l}+(-1)^{j_1+1}\underline{x_{i_1}^{-1}\cdots x_{i_{n-1}}^{-1}}x_{j_1}\frac{\partial F}{\partial x_{j_1}}\mathfrak{f}_{0j_1}\\&=(-1)^{j_1+1}\sum_{l=1}^{n-1}\underline{x_{i_1}^{-1}\cdots x_{i_l}^{0}\cdots x_{i_{n-1}}^{-1}}\left(\frac{\partial F}{\partial x_0}\mathfrak{f}_{i_lj_1}+\frac{\partial F}{\partial x_{j_1}}\mathfrak{f}_{0i_l}-\frac{\partial F}{\partial x_{i_l}}\mathfrak{f}_{0j_1}\right).\end{aligned}$$

Choose $c^{n-3,2}=(c^{n-3,2}_{i_1,\ldots,i_{n-2}})$ as

$$c^{n-3,2}_{i_1,\ldots,i_{n-2}}=(-1)^{\wp(\imath)}\underline{x_{i_1}^{-1}\cdots x_{i_{n-2}}^{-1}}\left(\frac{\partial F}{\partial x_0}\mathfrak{f}_{j_1j_2}-\frac{\partial F}{\partial x_{j_1}}\mathfrak{f}_{0j_2}+\frac{\partial F}{\partial x_{j_2}}\mathfrak{f}_{0j_1}\right)$$

which is again in $\ker\partial_{\boldsymbol{u}}$. Thus $d_{\mathcal{F}}(c^{n-3,2})=(-1)^{n-3}\partial_{\boldsymbol{v}}(c^{n-3,2})$ whose components are

$$\begin{aligned}d_{\mathcal{F}}(c^{n-3,2}_{i_1,\ldots,i_{n-2}})&=(-1)^{n-j_1-j_2}\Bigg(\underline{x_{i_1}^{-1}\cdots x_{i_{n-2}}^{-1}}x_0\frac{\partial F}{\partial x_0}\mathfrak{f}_{0j_1j_2}+\sum_{l=1}^{n-2}\underline{x_{i_1}^{-1}\cdots x_{i_l}^{0}\cdots x_{i_{n-2}}^{-1}}\frac{\partial F}{\partial x_0}\mathfrak{f}_{i_lj_1j_2}\\&\quad+\sum_{l=1}^{n-2}\underline{x_{i_1}^{-1}\cdots x_{i_l}^{0}\cdots x_{i_{n-2}}^{-1}}\frac{\partial F}{\partial x_{j_1}}\mathfrak{f}_{0i_lj_2}+\underline{x_{i_1}^{-1}\cdots x_{i_{n-2}}^{-1}}x_{j_1}\frac{\partial F}{\partial x_{j_1}}\mathfrak{f}_{0j_1j_2}\\&\quad+\sum_{l=1}^{n-2}\underline{x_{i_1}^{-1}\cdots x_{i_l}^{0}\cdots x_{i_{n-2}}^{-1}}\frac{\partial F}{\partial x_{j_2}}\mathfrak{f}_{0j_1i_l}+\underline{x_{i_1}^{-1}\cdots x_{i_{n-2}}^{-1}}x_{j_2}\frac{\partial F}{\partial x_{j_2}}\mathfrak{f}_{0j_1j_2}\Bigg)\\&=(-1)^{n-j_1-j_2}\sum_{l=1}^{n-2}\underline{x_{i_1}^{-1}\cdots x_{i_l}^{0}\cdots x_{i_{n-2}}^{-1}}\Bigg(\frac{\partial F}{\partial x_0}\mathfrak{f}_{i_lj_1j_2}+\frac{\partial F}{\partial x_{j_1}}\mathfrak{f}_{0i_lj_2}\\&\quad+\frac{\partial F}{\partial x_{j_2}}\mathfrak{f}_{0j_1i_l}-\frac{\partial F}{\partial x_{i_l}}\mathfrak{f}_{0j_1j_2}\Bigg).\end{aligned}$$

Choose $c^{n-4,3}=(c^{n-4,3}_{i_1,\ldots,i_{n-3}})$ as

$$c^{n-4,3}_{i_1,\ldots,i_{n-3}}=(-1)^{\wp(\imath)+1}\underline{x_{i_1}^{-1}\cdots x_{i_{n-3}}^{-1}}\left(\frac{\partial F}{\partial x_0}\mathfrak{f}_{j_1j_2j_3}-\frac{\partial F}{\partial x_{j_1}}\mathfrak{f}_{0j_2j_3}+\frac{\partial F}{\partial x_{j_2}}\mathfrak{f}_{0j_1j_3}-\frac{\partial F}{\partial x_{j_3}}\mathfrak{f}_{0j_1j_2}\right).$$

Set $j_0=0$ by convention and continue the above procedure. We obtain

$$c^{s-1,n-s}_{i_1,\ldots,i_s}=(-1)^{\wp(\imath)+n-s}\underline{x_{i_1}^{-1}\cdots x_{i_s}^{-1}}\sum_{m=0}^{n-s}(-1)^m\frac{\partial F}{\partial x_{j_m}}\mathfrak{f}_{j_0\cdots\widehat{j_m}\cdots j_{n-s}}\tag{5.4}$$

successively, which is obviously left preferred. In particular, when $s=1$,

$$c^{0,n-1}_{i_1}=(-1)^{n-i_1}\underline{x_{i_1}^{-1}}\sum_{m=0}^{n-1}(-1)^m\frac{\partial F}{\partial x_{j_m}}\mathfrak{f}_{j_0\cdots\widehat{j_m}\cdots j_{n-1}}$$

and hence

$$\begin{aligned}
d_{\mathcal{F}}(c_{i_1}^{0,n-1}) &= (-1)^{n-i_1}\left(\underline{x_{i_1}^0}\sum_{m=0}^{n-1}(-1)^m\frac{\partial F}{\partial x_{j_m}}\mathfrak{f}_{i_1 j_0\cdots\widehat{j_m}\cdots j_{n-1}} + \underline{x_{i_1}^{-1}}\sum_{m=0}^{n-1}x_{j_m}\frac{\partial F}{\partial x_{j_m}}\mathfrak{f}_{j_0\cdots j_{n-1}}\right)\\
&= (-1)^{n-i_1}\left(\underline{x_{i_1}^0}\sum_{m=0}^{n-1}(-1)^m\frac{\partial F}{\partial x_{j_m}}\mathfrak{f}_{i_1 j_0\cdots\widehat{j_m}\cdots j_{n-1}} - \underline{x_{i_1}^0}\frac{\partial F}{\partial x_{i_1}}\mathfrak{f}_{j_0\cdots j_{n-1}}\right)\\
&= (-1)^{n-1}\underline{x_{i_1}^0}\left(\sum_{j_m<i_1}(-1)^m\frac{\partial F}{\partial x_{j_m}}\mathfrak{f}_{j_0\cdots\widehat{j_m}\cdots i_1\cdots j_{n-1}} + \sum_{j_m>i_1}(-1)^{m+1}\frac{\partial F}{\partial x_{j_m}}\mathfrak{f}_{j_0\cdots i_1\cdots\widehat{j_m}\cdots j_{n-1}}\right.\\
&\quad\left. + (-1)^{i_1}\underline{x_{i_1}^0}\frac{\partial F}{\partial x_{i_1}}\mathfrak{f}_{j_0\cdots\widehat{i_1}\cdots j_{n-1}}\right)\\
&= (-1)^{n-1}\underline{x_{i_1}^0}\sum_{m=0}^{n}(-1)^m\frac{\partial F}{\partial x_{j_m}}\mathfrak{f}_{j_0\cdots\widehat{j_m}\cdots j_n}.
\end{aligned}$$

So $d_{\mathcal{F}}(c_{i_1}^{0,n-1})$ is actually the restriction of the global section $(-1)^{n-1}\boldsymbol{u}^\star$ to affine V_{i_1}. Hence the result follows. □

With minor modification, the proof of Lemma 5.4 is valid if the hypothesis $r \geq n$ is changed to $r < n$. Thus we obtain one more lemma as follows.

Lemma 5.5. *Suppose $d = n+1$ and $0 \leq r \leq n-1$. There exists a cocycle*

$$(0,\ldots,0,c^{n-1-r,r},c^{n-r,r-1},\ldots,c^{n-1,0})$$

in $\mathcal{H}_r^{n-1}$ where the components $c^{n-1-q,q}$ are given in (5.4). Each $c^{n-1-q,q}$ is left preferred in $\mathcal{H}_r^{n-1-q,q}$.

Note that there are n copies of k in the expression of $H^{n-1}(\mathcal{H}^\bullet)$. They respectively come from $\check{\mathbf{C}}^{n-1}(\mathfrak{U},\mathcal{F}_r^0)$ for $0 \leq r \leq n-1$. The class represented by the cocycle given in Lemma 5.5 is nontrivial since $c^{n-1,0}$ represents a nontrivial class. Consider the quasi-isomorphisms $\bar{\lambda}$ given in (4.2) and γ given in Theorem 4.4. The quasi-isomorphic image by $\gamma\bar{\lambda}\colon \mathcal{H}_r^\bullet \to \bar{\mathbf{C}}'^\bullet_{\mathrm{GS}}(\mathcal{A})_r$ is a collection of local sections of the sheaf $\wedge^r\mathcal{T}_X$. More precisely, we summarize the fact as

Proposition 5.6. *Suppose $d = n+1$. For every $0 \leq r \leq n-1$, there is a one-dimensional k-submodule of $H^{n-1-r}(X,\wedge^r\mathcal{T}_X)$, and consequently $H^{n-1-r}(X,\wedge^r\mathcal{T}_X) \neq 0$.*

5.3. **Case 3:** $d < n+1$. This is an easy case, since the complex (5.3) is zero. The results are

$${}^{II}E_{r,\infty}^{p,q} = {}^{II}E_{r,2}^{p,q} = \begin{cases} P_{r+(p-r)(d-1)}^{p-2r}, & r < p \leq 2r,\ q=0,\\ Q_r^{-r}, & p=r,\ q=0,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$H^i(\mathcal{H}^\bullet) \cong \bigoplus_{r<i} P_{r+(i-r)(d-1)}^{i-2r} \oplus Q_i^{-i}.$$

6. Applications

Based upon our computations in §5, we prove in §6.1 that a projective hypersurface is smooth if and only if the HKR decomposition of the second Hochschild cohomology group (1.5) holds (Theorem 6.3). This can be seen as an analogue of the characterization of smoothness of affine hypersurfaces (Remark 3.2). In the appendix A we give a less computational proof which works for complete intersections, which was suggested to us by the referee.

Recall that by definition of the GS complex, we have

$$\mathbf{C}^2_{\mathrm{GS}}(\mathcal{A}) = \mathbf{C}^{0,2}(\mathcal{A}) \oplus \mathbf{C}^{1,1}(\mathcal{A}) \oplus \mathbf{C}^{2,0}(\mathcal{A}).$$

We call a 2-cocycle $(m, f, c) \in \mathbf{C}^2_{\mathrm{GS}}(\mathcal{A})$ *untwined* (decomposable in [6]) if $(m, 0, 0)$, $(0, f, 0)$ and $(0, 0, c)$ are all 2-cocycles. A GS 2-class is called *intertwined* if it has no untwined representative (m, f, c). Intertwined classes are interesting from the point of view of deformation theory, as the only way to realize such a class is by simultanous non-trivial deformation of local multiplications and of restriction maps, with neither deforming only the multiplications, nor deforming only the restriction maps leading to a well-defined deformation. In §6.2, based upon the results from §5 we show that for a projective hypersurface as above if either $n \neq 2$ or $n = 2$ and $d \leq 4$, no intertwined 2-class exists. We give a family of concrete examples of intertwined 2-classes for $n = 2$ and $d \geq 6$.

Finally, in §6.3 we pay special attention to the case of quartic surfaces. We show that the dimension of $H^2_{\mathrm{GS}}(\mathcal{A})_1$ lies between 20 and 32, reaching all possible values except 30 and 31. The minimal value $H^2_{\mathrm{GS}}(\mathcal{A})_1 = 20$ is reached in the smooth K3 case. We also present an analysis of how $H^2_{\mathrm{GS}}(\mathcal{A})_1$ is built up from 2-classes of type $[(m, 0, 0)]$ and 2-classes of type $[(0, f, 0)]$, giving explicit computations in concrete examples.

6.1. **Characterization of smoothness.** In this section, we give a necessary and sufficient condition for a hypersurface to be smooth.

In the proof (not in the statement) of Theorem 6.3, we make use of the following subgroups of $H^2_{\mathrm{GS}}(\mathcal{A})_1$:

- the subgroup E_{res} of 2-classes of the form $[(0, f, 0)]$;
- the subgroup E_{mult} of 2-classes of the form $[(m, 0, 0)]$.

First of all, based upon the expression of $H^2(\mathcal{H}_1^\bullet)$ from §5, we obtain that $H^2_{\mathrm{GS}}(\mathcal{A})_1$ contains P^0_d as a summand for any n and d. Every element $t \in P^0_d$ corresponds to a class in E_{mult}. Let us consider when t also belongs to E_{res}.

Since $t \in P^0_d = (S/(\operatorname{im} \partial_{\boldsymbol{u}}))_d$, t lifts to an element $\bar{t}$ in S_d. We then identify $\bar{t}$ to a global section of $\mathcal{O}_X(d)$. For any $V \in \mathfrak{V}$, $\bar{t}|_V \in \mathcal{A}(V)$ determines the left multiplication by $\bar{t}|_V$ on $\mathcal{A}(V)$, and so $\bar{t}|_V \circ {}^\circ\mu$ represents a class in $H^2_{(1)}(\mathcal{A}(V), \mathcal{A}(V))$ which is independent of the choice of $\bar{t}$. Hence $t \in H^2_{\mathrm{GS}}(\mathcal{A})_1$ is represented by the GS 2-cocycle $(\bar{t} \circ {}^\circ\mu, 0, 0) := ((\bar{t}|_V \circ {}^\circ\mu)_V, 0, 0)$ which only deforms the local multiplications of $\mathcal{A}$. If $\bar{t}|_V \circ {}^\circ\mu$ happens to be a coboundary for all V, we have cochains $s_V \in C^1(\mathcal{A}(V), \mathcal{A}(V))$ such that $d_{\mathrm{Hoch}}(s_V) = \bar{t}|_V \circ {}^\circ\mu$. Let $s = (s_V)_V \in \bar{\mathbf{C}}'^{0,1}(\mathcal{A})$ and so $(\bar{t} \circ {}^\circ\mu, 0, 0) - (0, -d_{\mathrm{simp}}(s), 0) = d_{\mathrm{GS}}(s, 0)$. Thus $t = [(\bar{t} \circ {}^\circ\mu, 0, 0)] = [(0, -d_{\mathrm{simp}}(s), 0)]$ belongs to $E_{\mathrm{mult}} \cap E_{\mathrm{res}}$. In the other direction, if $t \in E_{\mathrm{mult}}$ is also in E_{res}, then we assume its representation is $(0, f, 0)$. The difference $(\bar{t} \circ {}^\circ\mu, 0, 0) - (0, f, 0)$ has to be a GS coboundary, say $d_{\mathrm{GS}}(s, 0)$. It follows that $\bar{t}|_V \circ {}^\circ\mu = d_{\mathrm{Hoch}}(s_V)$ for all $V \in \mathfrak{V}$.

Summarizing, $t \in E_{\mathrm{mult}} \cap E_{\mathrm{res}}$ if and only if $\bar{t}|_V \circ {}^\circ\mu$ is a Hochschild 2-coboundary for every $V \in \mathfrak{V}$. Note that $\mathcal{A}(V)$ is a localization of $\mathcal{A}(U)$ if $V \subseteq U$. It follows that $\bar{t}|_V \circ {}^\circ\mu$ is a coboundary of $\mathcal{A}(V)$ provided that $\bar{t}|_U \circ {}^\circ\mu$ is a coboundary of $\mathcal{A}(U)$. So this condition is again equivalent to the fact that $\bar{t}|_{U_i} \circ {}^\circ\mu$ is a coboundary of A_i for all $1 \leq i \leq n$. By §3,

$$H^2_{(1)}(A_i, A_i) = A_i \Big/ \left(\frac{\partial G_i}{\partial y_0}, \ldots, \frac{\partial G_i}{\partial y_{i-1}}, \frac{\partial G_i}{\partial y_{i+1}}, \ldots, \frac{\partial G_i}{\partial y_3} \right)$$

and $\bar{t}|_{U_i} \circ {}^\circ\mu$ is a coboundary if and only if $\bar{t}|_{U_i}$ is sent to zero by the projection $A_i \to H^2_{(1)}(A_i, A_i)$. Since $A_i = k[y_0 \ldots, y_{i-1}, y_{i+1}, \ldots, y_n]/(G_i)$ and

$$\sum_{j \neq i} y_j \frac{\partial G_i}{\partial y_j} + H_i = d \cdot G_i,$$

we have

$$H^2_{(1)}(A_i, A_i) = k[y_0, \dots, y_{i-1}, y_{i+1}, \dots, y_n] \Big/ \Big(\frac{\partial G_i}{\partial y_0}, \dots, \frac{\partial G_i}{\partial y_{i-1}}, H_i, \frac{\partial G_i}{\partial y_{i+1}}, \dots, \frac{\partial G_i}{\partial y_n} \Big).$$

Recall the definition of H_i given in §4.1. There is an algebra map $P^0 \to H^2_{(1)}(A_i, A_i)$ defined by $x_j \mapsto y_j$ if $j \neq i$ and $x_i \mapsto 1$, whose kernel is $(x_i - 1)P^0$. Thus $t \in E_{\mathrm{res}}$ if and only if $t \in \cap_{i=1}^n (x_i - 1)P^0$. Notice that t is homogeneous. If $t = (1 - x_i)T_i$ for some $T_i \in P^0$, by comparing the homogeneous components, we conclude that t is annihilated by a power of x_i and so $T_i = \sum_{m=0}^\infty t x_i^m$ is actually a finite sum. In the opposite direction, if t is annihilated by a power of x_i, then $t = (1 - x_i) \sum_{m=0}^\infty t x_i^m \in (x_i - 1)P^0$. Consequently, we have proven

Lemma 6.1. *Let $t \in P^0_d$. Then $t \in E_{\mathrm{res}}$ if and only if $x_i \in \sqrt{\mathrm{ann}_{P^0}(t)}$ for all $1 \leq i \leq n$.*

Next let us recall the work [9] by Gerstenhaber and Schack. Starting from their Hodge decomposition for presheaves of commutative algebras

$$H^i_{\mathrm{GS}}(\mathcal{A}) = \bigoplus_{r \in \mathbb{N}} H^i_{\mathrm{GS}}(\mathcal{A})_r, \tag{6.1}$$

they prove the existence of the HKR type decomposition

$$H^i_{\mathrm{GS}}(\mathcal{A}) \cong \bigoplus_{p+q=i} H^p_{\mathrm{simp}}(\mathfrak{V}, \wedge^q \mathcal{T})$$

for any smooth complex projective variety X, where $\mathcal{A} = \mathcal{O}_X|_{\mathfrak{V}}$ (resp. $\mathcal{T} = \mathcal{T}_X|_{\mathfrak{V}}$) is the restriction of the structure sheaf (resp. tangent sheaf) to an affine open covering $\mathfrak{V}$ closed under intersection. In particular,

$$H^2_{\mathrm{GS}}(\mathcal{A}) \cong H^0_{\mathrm{simp}}(\mathfrak{V}, \wedge^2 \mathcal{T}) \oplus H^1_{\mathrm{simp}}(\mathfrak{V}, \mathcal{T}) \oplus H^2_{\mathrm{simp}}(\mathfrak{V}, \mathcal{A}).$$

The roles played by the three summands in the deformation of $\mathcal{A}$ (viewed as a twisted presheaf) are explained in [6]. More concretely, elements in the three summands respectively deform the (local) multiplications, the restriction maps, and the twisting elements of $\mathcal{A}$. If X is not necessarily smooth, Gerstenhaber and Schack's result remains partially correct: $H^i_{\mathrm{GS}}(\mathcal{A})_r \cong H^{i-r}_{\mathrm{simp}}(\mathfrak{V}, \wedge^r \mathcal{T})$ if $r = 0$ or $r = i$, and in general $H^i_{\mathrm{GS}}(\mathcal{A})_{i-1}$ contains $H^1_{\mathrm{simp}}(\mathfrak{V}, \wedge^{i-1} \mathcal{T})$ as a k-submodule. For $i = 2$, we more precisely have

$$H^1_{\mathrm{simp}}(\mathfrak{V}, \mathcal{T}) \cong E_{\mathrm{res}} \subseteq H^2_{\mathrm{GS}}(\mathcal{A})_1. \tag{6.2}$$

In particular, (6.1) now yields

$$H^2_{\mathrm{GS}}(\mathcal{A}) \cong H^0_{\mathrm{simp}}(\mathfrak{V}, \wedge^2 \mathcal{T}) \oplus H^1_{\mathrm{simp}}(\mathfrak{V}, \mathcal{T}) \oplus H^2_{\mathrm{simp}}(\mathfrak{V}, \mathcal{A}) \oplus E. \tag{6.3}$$

where E is a complement of E_{res} in $H^2_{\mathrm{GS}}(\mathcal{A})_1$.

When X is a projective hypersurface, the isomorphism $H^p(X, \wedge^q \mathcal{T}_X) \cong H^p_{\mathrm{simp}}(\mathfrak{V}, \wedge^q \mathcal{T})$ holds for all p, q. The decomposition (6.3) is equivalent to

$$HH^2(X) \cong H^0(X, \wedge^2 \mathcal{T}_X) \oplus H^1(X, \mathcal{T}_X) \oplus H^2(X, \mathcal{O}_X) \oplus E.$$

We have thus proven:

Proposition 6.2. *Let X be a projective hypersurface. The following are equivalent:*

(1) *The HKR decomposition holds for the second cohomology, i.e.*

$$HH^2(X) \cong H^0(X, \wedge^2 \mathcal{T}_X) \oplus H^1(X, \mathcal{T}_X) \oplus H^2(X, \mathcal{O}_X).$$

(2) *We have $H^1(X, \mathcal{T}_X) \cong E_{\mathrm{res}} = H^2_{\mathrm{GS}}(\mathcal{A})_1$.*

Remark 6.1. In deformation theoretic terms, Proposition 6.2 states that for a projective hypersurface X, the HKR decomposition holds for $HH^2(X)$ if and only if every (commutative) scheme deformation of X can be realized by only deforming restriction maps while trivially deforming individual algebras on an affine cover. This is the classical deformation picture for smooth schemes.

We have the following converse of the HKR theorem for projective hypersurfaces:

Theorem 6.3. *Let X be a projective hypersurface. The following are equivalent:*

(1) *X is smooth.*
(2) *The HKR decomposition holds for all cohomology groups, i.e.*

$$HH^i(X) \cong \bigoplus_{p+q=i} H^p(X, \wedge^q \mathcal{T}_X), \quad \forall\, i \in \mathbb{N}.$$

(3) *The HKR decomposition holds for the second cohomology, i.e.*

$$HH^2(X) \cong H^0(X, \wedge^2 \mathcal{T}_X) \oplus H^1(X, \mathcal{T}_X) \oplus H^2(X, \mathcal{O}_X).$$

Proof. It remains to prove (3) ⇒ (1). Assume X is a hypersurface of degree d in $\mathbb{P}^n$ which is not smooth. According to Proposition 6.2, it suffices to produce a class in $H^2_{\mathrm{GS}}(\mathcal{A})_1 \setminus E_{\mathrm{res}}$. At least one of the algebras A_i is not smooth, say A_n. It follows from Remark 3.2 that $H^2_{(1)}(A_n, A_n) \neq 0$. As before, we know

$$\begin{aligned} H^2_{(1)}(A_n, A_n) &= k[y_0, \ldots, y_{n-1}] \Big/ \left(\frac{\partial G_n}{\partial y_0}, \cdots, \frac{\partial G_n}{\partial y_{n-1}}, H_n \right) \\ &\cong R \Big/ \left(x_n - 1, \frac{\partial F}{\partial x_0}, \cdots, \frac{\partial F}{\partial x_{n-1}}, \frac{\partial F}{\partial x_n} \right) \\ &= P^0/(x_n - 1). \end{aligned}$$

Since $P^0/(x_n-1) \neq 0$ this implies that $0 \neq x_n^m \in P^0$ for any $m \in \mathbb{N}$. In particular, $0 \neq x_n^d \in P^0_d$ presents a non-trivial class in $H^2_{\mathrm{GS}}(\mathcal{A})_1$, and $x_n \notin \sqrt{\mathrm{ann}_{P^0}(x_n^d)}$. By Lemma 6.1, $x_n^d \notin E_{\mathrm{res}}$, which finishes the proof. □

Remark 6.2. The inverse HKR result formulated in Theorem 6.3 actually holds true in greater generality, and a proof for complete intersections based upon global generation of the normal sheaf, which was suggested to us by the referee, is presented in the appendix A.

However, our original computational proof based upon Lemma 6.1, in which the idea is to catch a deformation in an affine piece that can be lifted to a global one, may be of independent value. In particular, later on we apply this idea in order to determine efficiently whether a class in E_{mult} belongs to E_{res} (See Table 1).

6.2. **Examples of intertwined classes.** We are particularly interested in $HH^2(X)$ since it parameterizes the equivalence classes of first order deformations of X. We retain the notations used before. On one hand, we have the decomposition (6.3). On the other hand, any GS 2-cocycle

$$(m, f, c) \in \bar{\mathbf{C}}'^{0,2}(\mathcal{A}) \oplus \bar{\mathbf{C}}'^{1,1}(\mathcal{A}) \oplus \bar{\mathbf{C}}'^{2,0}(\mathcal{A})$$

factors as $(m - m^{\mathrm{ab}}, 0, 0) + (m^{\mathrm{ab}}, f, 0) + (0, 0, c)$ under the Hodge decomposition where m^{ab} depends only on m. Since $E \subseteq H^2_{\mathrm{GS}}(\mathcal{A})_1$, the elements in E admit representatives of the form $(m, f, 0)$. Normally, neither $(m, 0, 0)$ nor $(0, f, 0)$ is a cocycle. The cocycle is called *untwined* if $(m, 0, 0)$ or, equivalently $(0, f, 0)$ is a cocycle. A 2-class is called *intertwined* if it has no untwined representative.

In this section, we will given examples of such intertwined 2-classes. By the decomposition of $\mathcal{H}^\bullet$ and by Theorem 4.4, classes in $H^2(\mathcal{H}^\bullet_0)$ and $H^2(\mathcal{H}^\bullet_2)$ have untwined representatives of the form $(0, 0, c)$ and $(m, 0, 0)$ respectively. It is sufficient to consider $H^2(\mathcal{H}^\bullet_1)$.

First of all, by the discussion in §5, $H^2(\mathcal{H}_1^\bullet)$ is the direct sum of P_d^0 and Q_2^{-2} if $d < n+1$. Via the quasi-isomorphisms $\mathcal{H}^\bullet \to \mathcal{G}^\bullet \to \mathcal{E}^\bullet \to \bar{\mathbf{C}}'^\bullet_{\mathrm{GS}}(\mathcal{A})$, any element in P_d^0 or Q_2^{-2} gives rise to a GS 2-class of the form $[(m,0,0)] \in H^2_{\mathrm{GS}}(\mathcal{A})$. So intertwined 2-class never exists if $d < n+1$.

Next, besides P_d^0 and Q_2^{-2}, $H^2(\mathcal{H}_1^\bullet)$ contains k as a direct summand if $d = n+1$. By Proposition 5.6, any nonzero element in k corresponds to a nonzero class in $H^1(X, \mathcal{T}_X)$ which clearly admits a representative of the form $(0, f, 0)$.

Thus an intertwined class exists only possibly in $\mathscr{S}(Z_{d-4}^{n-3})$ in the case $d > n+1$. Necessarily, $n \le 3$ since $Z^{n-3} = 0$ for all $n > 3$. Since $n = 3$ implies $\mathscr{S}(Z_{d-4}^0) \subseteq H^2(\mathcal{H}_0^\bullet)$, $n = 2$ is the unique choice, and so $d > 3$. Moreover, by the definition of Z_{d-4}^{-1}, the short sequence

$$0 \longrightarrow Z_{d-4}^{-1} \longrightarrow R_{d-4}^3 \xrightarrow{\partial_{\boldsymbol{v}}} R_{d-3} \longrightarrow 0 \tag{6.4}$$

is exact. It follows that $Z_{d-4}^{-1} \neq 0$ only if $d > 4$.

We have proven:

Proposition 6.4. *Suppose either $n \neq 2$ or $n = 2$ and $d \le 4$. Then $H^2_{\mathrm{GS}}(\mathcal{A})$ does not contain an intertwined cohomology class.*

Now let $d \ge 6$ and $F = x_0^d + x_1^{d-1}x_2$. The map $\partial_{\boldsymbol{v}} \colon R_1^3 \to R_2$ in (6.4) sends (r_0, r_1, r_2) to $r_0x_0 + r_1x_1 + r_2x_2$, whose kernel is 3-dimensional with a basis $\{(-x_1, x_0, 0), (-x_2, 0, x_0), (0, -x_2, x_1)\}$. Since $\mathscr{S}(Z_1^{-1})$ arises from $\mathcal{H}_1^\bullet$, we consider the double complex

$$\begin{array}{ccccc}
\underline{S_{x_1} \oplus S_{x_2}} & \longrightarrow & S_{x_1x_2} & & \\
\uparrow{\scriptstyle \partial_{\boldsymbol{u}}} & & \uparrow{\scriptstyle \partial_{\boldsymbol{u}}} & & \\
S^3_{x_1} \oplus S^3_{x_2} & \longrightarrow & \underline{S^3_{x_1x_2}} & & \\
\uparrow{\scriptstyle \partial_{\boldsymbol{v}}} & & \uparrow{\scriptstyle \partial_{\boldsymbol{v}}} & & \\
S_{x_1} \oplus S_{x_2} & \longrightarrow & S_{x_1x_2} & \longrightarrow & \underline{0}
\end{array}$$

with three entries corresponding to $\mathcal{H}_1^2$ underlined. We choose the basis element $(0, -x_2, x_1)$, and so

$$\mathscr{S}(0, -x_2, x_1) = (0, -x_0^4x_1^{-1}x_2^{-2}, x_0^4x_1^{-2}x_2^{-1}) \in S^3_{x_1x_2}.$$

Since $\boldsymbol{u} = (dx_0^{d-1}, (d-1)x_1^{d-2}x_2, x_1^{d-1})$, $\partial_{\boldsymbol{u}}(\mathscr{S}(0, -x_2, x_1))$ is equal to

$$(d-1)x_1^{d-2}x_2 \cdot (-x_0^4x_1^{-1}x_2^{-2}) + x_1^{d-1} \cdot x_0^4x_1^{-2}x_2^{-1} = -(d-2)x_0^4x_1^{d-3}x_2^{-1}.$$

Choose $(0, (d-2)x_0^4x_1^{d-3}x_2^{-1}) \in S_{x_1} \oplus S_{x_2}$, and thus $((0, (d-2)x_0^4x_1^{d-3}x_2^{-1}), \mathscr{S}(0, -x_2, x_1), 0)$ is a 2-cocycle in $\mathcal{H}_1^\bullet$.

Let us prove that the class $\mathfrak{c} := [((0, (d-2)x_0^4x_1^{d-3}x_2^{-1}), \mathscr{S}(0, -x_2, x_1), 0)]$ is intertwined. Assume it can be written as $[(m', 0, 0)] + [(0, f', 0)]$, then $m' := (m_1', m_2') \in \ker\{S_{x_1} \oplus S_{x_2} \to S_{x_1x_2}\}$. Note that S, S_{x_1} and S_{x_2} can be regarded as k-submodules of $S_{x_1x_1}$ since S is a domain, and that $S_{x_1} \cap S_{x_2} = S$. We then have $m_1' = m_2'$ and so $m_2' \in S$. It follows that $m_2' + (d-2)x_0^4x_1^{d-3}x_2^{-1} \in \operatorname{im}\{\partial_{\boldsymbol{u}} \colon S^3_{x_2} \to S_{x_2}\}$, say

$$m_2' + (d-2)x_0^4x_1^{d-3}x_2^{-1} = dx_0^{d-1}a_1 + (d-1)x_1^{d-2}x_2a_2 + x_1^{d-1}a_3 \tag{6.5}$$

for some a_1, a_2, $a_3 \in S_{x_2}$. By considering their degrees, we have

$$a_1 = \sum_{\substack{0 \le i_0 < d \\ i_1 \ge 0}} \lambda_1^{i_0i_1} x_0^{i_0} x_1^{i_1} x_2^{1-i_0-i_1}$$

and similarly for a_2, a_3. The right-hand side of (6.5) is

$$\sum_{i_1\geq 0} d\lambda_1^{0i_1} x_0^{d-1} x_1^{i_1} x_2^{1-i_1} - \sum_{\substack{1\leq i_0<d\\ i_1\geq 0}} d\lambda_1^{i_0 i_1} x_0^{i_0-1} x_1^{d+i_1-1} x_2^{2-i_0-i_1}$$
$$+ \sum_{\substack{0\leq i_0<d\\ i_1\geq 0}} (d-1)\lambda_2^{i_0 i_1} x_0^{i_0} x_1^{d+i_1-2} x_2^{2-i_0-i_1} + \sum_{\substack{0\leq i_0<d\\ i_1\geq 0}} \lambda_3^{i_0 i_1} x_0^{i_0} x_1^{d+i_1-1} x_2^{1-i_0-i_1}.$$

Observe that the basis element $x_0^4 x_1^{d-3} x_2^{-1}$ never appears in any term of the right-hand side, since $d \geq 6$ and $i_1 \geq 0$. Together with the fact $m_2' \in S$, we get a contradiction. Thus $\mathfrak{c}$ is indeed an intertwined class.

We remind the reader that the projective curve $x_0^d + x_1^{d-1}x_2$ has a unique singularity $(0:0:1)$.

Next let us describe how the class deforms $\mathcal{A}$ in the case $d = 6$. We have $\mathfrak{U} = \{U_1, U_2\}$ and $\mathfrak{V} = \{V_1, V_2, V_{12}\}$, and define $\lambda\colon \mathfrak{V} \to \mathfrak{U}$ by

$$V_1 \mapsto U_1, \quad V_2 \mapsto U_2, \quad V_{12} \mapsto U_2.$$

The algebras A_1, A_2, A_{12} are expressed as $k[y_0, y_2]/(y_0^6+y_2)$, $k[y_0, y_1]/(y_0^6+y_1^5)$, $k[y_0, y_1, y_1^{-1}]/(y_0^6+y_1^5)$ respectively. By the formula (4.2), we obtain a 2-cocycle $(e^0, e^1, 0)$ in $\mathcal{E}_1^\bullet$ given by

$$\begin{aligned}
e^0_{V_1} &= 0,\\
e^0_{V_2} &= -4x_0^5x_1^3x_2^{-1}|_{V_2} = -4y_0^5y_1^3 \in A_2,\\
e^0_{V_{12}} &= -4y_0^5y_1^3 \in A_{12},\\
e^1_{V_{12}\subset V_1} &= -(0, -x_0^4x_1^{-1}x_2^{-2}, x_0^4x_1^{-2}x_2^{-1})|_{V_{12}} = (0, y_0^4y_1^{-1}, y_0^4y_1^{-2}) \in A_{12}^3,\\
e^1_{V_{12}\subset V_2} &= 0.
\end{aligned}$$

So by Theorem 4.4, the intertwined cocycle $(m, f, 0)$ is given by

$$\begin{aligned}
m_{V_2} &= -4y_0^5y_1^{3\circ}\mu_{A_2},\\
m_{V_{12}} &= -4y_0^5y_1^{3\circ}\mu_{A_{12}},\\
f_{V_{12}\subset V_1} &= \left(-y_0^4y_1^{-2}\frac{{}^\circ\partial}{{}^\circ\partial y_0} + (y_0^4y_1^{-1} - y_0^4y_1^{-2})\frac{{}^\circ\partial}{{}^\circ\partial y_1}\right)\circ \rho^{V_1}_{V_{12}}
\end{aligned}$$

and other components equal to zero, where $\rho^{V_1}_{V_{12}}\colon A_1 \to A_{12}$ is the restriction map.

Unfortunately, the authors have not found any intertwined class in the case $d = 5$. So we pose the following open question:

Question: Does an intertwined 2-class exist for a degree 5 curve in $\mathbb{P}^2$?

6.3. **The second cohomology groups of quartic surfaces.** As we exhibited in §6.2, intertwined 2-classes exist for some non-smooth curves. In contrast, by Proposition 6.4 such classes do not exist for higher dimensional hypersurfaces, whence for these it suffices to study 2-cocycles of the form $(m, 0, 0)$, $(0, f, 0)$ and $(0, 0, c)$ separately. Among projective hypersurfaces, we are particularly interested in quartic surfaces in $\mathbb{P}^3$.

From now on, let X be a projective quartic surface in $\mathbb{P}^3$, i.e. $n = 3$ and $d = 4$. By the discussion in §5,

$$\begin{aligned}
H^2_{\mathrm{GS}}(\mathcal{A})_0 &\cong k;\\
H^2_{\mathrm{GS}}(\mathcal{A})_1 &\cong k \oplus P_4^0;\\
H^2_{\mathrm{GS}}(\mathcal{A})_2 &\cong k \oplus Q_2^{-2}.
\end{aligned}$$

Now let us make the three deformations arising from the three components "k" above explicit, following Lemma 5.5 and formula (5.4). A direct computation shows that

$$\begin{aligned}
c_{123}^{2,0} &= x_1^{-1}x_2^{-1}x_3^{-1}\frac{\partial F}{\partial x_0},\\
c_{12}^{1,1} &= x_1^{-1}x_2^{-1}\Big(\frac{\partial F}{\partial x_3}\mathfrak{f}_0 - \frac{\partial F}{\partial x_0}\mathfrak{f}_3\Big),\\
c_{13}^{1,1} &= x_1^{-1}x_3^{-1}\Big(-\frac{\partial F}{\partial x_2}\mathfrak{f}_0 + \frac{\partial F}{\partial x_0}\mathfrak{f}_2\Big),\\
c_{23}^{1,1} &= x_2^{-1}x_3^{-1}\Big(\frac{\partial F}{\partial x_1}\mathfrak{f}_0 - \frac{\partial F}{\partial x_1}\mathfrak{f}_3\Big),\\
c_{1}^{0,2} &= x_1^{-1}\Big(\frac{\partial F}{\partial x_3}\mathfrak{f}_{02} - \frac{\partial F}{\partial x_2}\mathfrak{f}_{03} + \frac{\partial F}{\partial x_0}\mathfrak{f}_{23}\Big),\\
c_{2}^{0,2} &= x_2^{-1}\Big(-\frac{\partial F}{\partial x_3}\mathfrak{f}_{01} + \frac{\partial F}{\partial x_2}\mathfrak{f}_{03} - \frac{\partial F}{\partial x_0}\mathfrak{f}_{23}\Big),\\
c_{3}^{0,2} &= x_3^{-1}\Big(\frac{\partial F}{\partial x_2}\mathfrak{f}_{01} - \frac{\partial F}{\partial x_1}\mathfrak{f}_{02} + \frac{\partial F}{\partial x_0}\mathfrak{f}_{12}\Big).
\end{aligned}$$

We choose a map $\lambda\colon \mathcal{V} \to \mathcal{U}$ by $\lambda(V_{j_1\dots j_r}) = U_{j_r}$ if $j_1 < \cdots < j_r$, and the algebra $\mathcal{A}(V_{j_1\dots j_r})$ is expressed as $k[y_0,\dots,y_{j_r-1},y_{j_r+1},\dots,y_3,y_{j_1}^{-1},\dots,y_{j_{r-1}}^{-1}]/(G_{j_r})$. By (4.2), $c^{2,0}$ gives rise to a 2-cocycle $(0,0,e^2)$ in $\mathcal{E}_0$ by

$$e^2_{V_{123}\subset V_{12}\subset V_1} = -x_1^{-1}x_2^{-1}x_3^{-1}\frac{\partial F}{\partial x_0}\Big|_{V_{123}} = -y_1^{-1}y_2^{-1}\frac{\partial G_3}{\partial y_0}.$$

This in turn gives rise to the GS cocycle $(0,0,c)$ by

$$c_{V_{123}\subset V_{12}\subset V_1} = -y_1^{-1}y_2^{-1}\frac{\partial G_3}{\partial y_0}.$$

Using (4.2) again, we obtain a 2-cocycle $(0,e^1,e^2)$ in $\mathcal{E}_1$ from $(0,c^{1,1},c^{2,0})$ with e^2 as above and e^1 given by

$$\begin{aligned}
e^1_{V_{12}\subset V_1} &= y_1^{-1}\Big(-\frac{\partial G_2}{\partial y_3}\mathfrak{f}_0 + \frac{\partial G_2}{\partial y_0}\mathfrak{f}_3\Big), & e^1_{V_{123}\subset V_1} &= y_1^{-1}\Big(-\frac{\partial G_3}{\partial y_2}\mathfrak{f}_0 + \frac{\partial G_3}{\partial y_0}\mathfrak{f}_2\Big),\\
e^1_{V_{13}\subset V_1} &= y_1^{-1}\Big(\frac{\partial G_3}{\partial y_2}\mathfrak{f}_0 - \frac{\partial G_3}{\partial y_0}\mathfrak{f}_2\Big), & e^1_{V_{123}\subset V_2} &= y_2^{-1}\Big(-\frac{\partial G_3}{\partial y_1}\mathfrak{f}_0 + \frac{\partial G_3}{\partial y_1}\mathfrak{f}_3\Big),\\
e^1_{V_{23}\subset V_2} &= y_2^{-1}\Big(-\frac{\partial G_3}{\partial y_1}\mathfrak{f}_0 + \frac{\partial G_3}{\partial y_1}\mathfrak{f}_3\Big), & e^1_{V_{123}\subset V_{12}} &= y_2^{-1}\Big(-\frac{\partial G_3}{\partial y_1}\mathfrak{f}_0 + \frac{\partial G_3}{\partial y_1}\mathfrak{f}_3\Big).
\end{aligned}$$

Then we can deduce a GS cocycle $(0,f,0)$ from $(0,e^1,e^2)$. Notice that the expression of m is independent of e^2. To have the expression explicitly, by the discussion in §2, we only have to replace the formal base element $\mathfrak{f}_i$ by $^\circ\partial/^\circ\partial y_i$, then compose with the restriction map. For example,

$$m_{V_{12}\subset V_1} = y_1^{-1}\Big(-\frac{\partial G_2}{\partial y_3}\frac{^\circ\partial}{^\circ\partial y_0} + \frac{\partial G_2}{\partial y_0}\frac{^\circ\partial}{^\circ\partial y_3}\Big)\circ\rho^{V_1}_{V_{12}},$$

and so on. Likewise, we conclude that the cocycle (e^0,e^1,e^2) in $\mathcal{E}_2$ induced by $(c^{0,2},c^{1,1},c^{2,0})$ has the form

$$\begin{aligned}
e^0_{V_1} &= \frac{\partial G_1}{\partial y_3}\mathfrak{f}_{02} - \frac{\partial G_1}{\partial y_2}\mathfrak{f}_{03} + \frac{\partial G_1}{\partial y_0}\mathfrak{f}_{23},\\
e^0_{V_2} &= -\frac{\partial G_2}{\partial y_3}\mathfrak{f}_{01} + \frac{\partial G_2}{\partial y_2}\mathfrak{f}_{03} - \frac{\partial G_2}{\partial y_0}\mathfrak{f}_{13},\\
e^0_{V_3} &= \frac{\partial G_3}{\partial y_2}\mathfrak{f}_{01} - \frac{\partial G_3}{\partial y_1}\mathfrak{f}_{02} + \frac{\partial G_3}{\partial y_0}\mathfrak{f}_{12}.
\end{aligned}$$

Thus (e^0,e^1,e^2) induces the GS cocycle $(m,0,0)$ given by

$$m_{V_1} = \frac{\partial G_1}{\partial y_3}\cdot\frac{^\circ\partial}{\partial y_0}\cup\frac{^\circ\partial}{\partial y_2} - \frac{\partial G_1}{\partial y_2}\cdot\frac{^\circ\partial}{\partial y_0}\cup\frac{^\circ\partial}{\partial y_3} + \frac{\partial G_1}{\partial y_0}\cdot\frac{^\circ\partial}{\partial y_2}\cup\frac{^\circ\partial}{\partial y_3},$$

$$
\begin{aligned}
(6.6)\qquad m_{V_2} &= -\frac{\partial G_2}{\partial y_3}\cdot\frac{^\circ\partial}{\partial y_0}\cup\frac{^\circ\partial}{\partial y_1}+\frac{\partial G_2}{\partial y_2}\cdot\frac{^\circ\partial}{\partial y_0}\cup\frac{^\circ\partial}{\partial y_3}+\frac{\partial G_2}{\partial y_2}\cdot\frac{^\circ\partial}{\partial y_1}\cup\frac{^\circ\partial}{\partial y_3},\\
m_{V_3} &= \frac{\partial G_3}{\partial y_2}\cdot\frac{^\circ\partial}{\partial y_0}\cup\frac{^\circ\partial}{\partial y_1}-\frac{\partial G_3}{\partial y_1}\cdot\frac{^\circ\partial}{\partial y_0}\cup\frac{^\circ\partial}{\partial y_2}+\frac{\partial G_3}{\partial y_0}\cdot\frac{^\circ\partial}{\partial y_1}\cup\frac{^\circ\partial}{\partial y_2}.
\end{aligned}
$$

Let us look into the dimensions of $H^2_{\mathrm{GS}}(\mathcal{A})_r$ for $r = 0, 1, 2$. Obviously, $\dim H^2_{\mathrm{GS}}(\mathcal{A})_0 = 1$. Since $P^0_4 = (S/(\operatorname{im}\partial_{\boldsymbol{u}}))_4 = (R/(\operatorname{im}\partial_{\boldsymbol{u}}))_4 = R_4/\sum_{i,j=0}^3 kx_i\cdot\partial F/\partial x_j$, we have the following inequality

$$\dim P^0_4 = \dim R_4 - \dim\sum_{i,j=0}^3 x_i\frac{\partial F}{\partial x_j} = 35 - \dim\sum_{i,j=0}^3 kx_i\frac{\partial F}{\partial x_j} \geq 35-16 = 19.$$

Next we investigate the upper bound of $\dim P^0_4$. Obviously, $\{x_i\cdot\partial F/\partial x_j\}_{0\leq i\leq 3}$ is k-linearly independent provided that $\partial F/\partial x_j \neq 0$. In particular, $\dim\sum_{i=0}^3 kx_i\cdot\partial F/\partial x_0 = 4$ and hence $\dim P^0_4 \leq 31$. Interestingly, there is a gap between 31 and other possible dimensions. Let us prove

Lemma 6.5. *If* $\dim P^0_4 \neq 31$*, then* $19 \leq \dim P^0_4 \leq 28$.

Proof. Suppose $F = x_0^4 + f_1x_0^3 + f_2x_0^2 + f_3x_0 + f_4$ where $f_t \in k[x_1, x_2, x_3]$ are homogeneous of degree t.

First of all, let us reduce the lemma to the case $f_1 = 0$. In fact, $\dim P^0_4 = \dim H^2_{\mathrm{GS}}(\mathcal{A})_1 - 1$ is invariant under isomorphism of surfaces. By an argument similar to the argument presented in the paragraph after Theorem 4.4, f_1 can be annihilated via the isomorphism

$$x_0 \mapsto x_0 - \frac{1}{4}f_1,\quad x_j \mapsto x_j\quad (j = 1, 2, 3).$$

Now we safely assume $f_1 = 0$. Since $\dim P^0_4 \neq 31$, one of $\partial F/\partial x_1$, $\partial F/\partial x_2$, $\partial F/\partial x_3$ is nonzero, say $\partial F/\partial x_1 \neq 0$. By comparing the degrees of $\partial F/\partial x_0$ and $\partial F/\partial x_1$ with respect to x_0, we obtain

$$(\lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3)\frac{\partial F}{\partial x_1} \in \sum_{l=0}^3 kx_l\frac{\partial F}{\partial x_0}$$

for some $\lambda_1, \lambda_2, \lambda_3 \in k$ only when $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Hence

$$\sum_{i,j=0}^3 kx_i\frac{\partial F}{\partial x_j} \supseteq \sum_{i=0}^3 kx_i\frac{\partial F}{\partial x_0} + \sum_{i=1}^3 kx_i\frac{\partial F}{\partial x_1} = \bigoplus_{i=0}^3 kx_i\frac{\partial F}{\partial x_0} \oplus \bigoplus_{i=1}^3 kx_i\frac{\partial F}{\partial x_1} \cong k^7.$$

It follows that $\dim P^0_4 \leq 35 - 7 = 28$. □

Therefore, $\dim H^2_{\mathrm{GS}}(\mathcal{A})_1 \in \{20, \dots, 29\} \cup \{32\}$. The dimension indeed reaches every number in the set. We list some examples in Table 1 showing this fact. By Lemma 6.1, we are able to check if $t \in P^0_4$ also corresponds to a class in $H^1(X, \mathcal{T}_X)$. Accordingly, the dimensions of $H^1(X, \mathcal{T}_X)$ for these examples can be computed, as listed in the third column.

For $r = 2$, the group Q^{-2}_2 comes from the complex

$$S^6_2/\operatorname{im}\partial_{\boldsymbol{v}} \xrightarrow{\partial_{\boldsymbol{u}}} S^4_5/\operatorname{im}\partial_{\boldsymbol{v}} \xrightarrow{\partial_{\boldsymbol{u}}} S_8$$

by (5.1). It fits into a projection

$$
\begin{array}{ccccc}
R^6_2 & \xrightarrow{\ \partial_{\boldsymbol{u}}\ } & R^4_5 & \xrightarrow{\ \partial_{\boldsymbol{u}}\ } & R_8\\
\downarrow & & \downarrow & & \downarrow\\
S^6_2/\operatorname{im}\partial_{\boldsymbol{v}} & \xrightarrow{\ \partial_{\boldsymbol{u}}\ } & S^4_5/\operatorname{im}\partial_{\boldsymbol{v}} & \xrightarrow{\ \partial_{\boldsymbol{u}}\ } & S_8
\end{array}
$$

of complexes. By Euler's formula, the projection turns out to be a quasi-isomorphism. Hence $Q^{-2}_2 \cong \ker\{\partial_{\boldsymbol{u}}\colon R^6_2 \to R^4_5\}$. The dimension of the latter is easier to compute than that of Q^{-2}_2. Let elements in R^6_2 be expressed by

$$(a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{13}).$$

F	$\dim H^2_{\mathrm{GS}}(\mathcal{A})_1$	$\dim H^1(X, \mathcal{T}_X)$	$\dim H^2_{\mathrm{GS}}(\mathcal{A})_2$
$x_0^4 + x_1^4 + x_2^4 + x_3^4$	20	20	1
$(x_0^2 + x_1^2)^2 + x_2^4 + x_3^4$	21	4	1
$(x_0^2 + x_1^2)^2 + (x_2^2 + x_3^2)^2$	22	2	2
$(x_0^2 + x_1^2 + x_2^2)^2 + x_3^4$	23	2	5
$x_0^4 + x_1^4 + x_2^4$	24	1	1
$(x_0^2 + x_1^2)^2 + x_2^4$	25	1	5
$(x_0^2 + x_1^2 + x_2^2 + x_3^2)^2$	26	1	17
$(x_0^2 + x_1^2 + x_2^2)^2$	27	1	17
$x_0^4 + x_1^4$	28	1	11
$(x_0^2 + x_1^2)^2$	29	1	11
x_0^4	32	1	31

TABLE 1. dimensions of several groups

If $F = x_0^4 + (x_1^2 + x_2^2)^2$, then

$$\ker\{\partial_{\boldsymbol{u}} \colon R_2^6 \to R_5^4\} = \{(0, 0, 0, 0, x_2 u, -x_1 u) \mid u \in R_1\}$$

and hence Q_2^{-2} is equal to

$$\{(0, 0, 0, 0, x_2 u, -x_1 u) + \operatorname{im} \partial_{\boldsymbol{v}} \mid u \in S_1\}$$

whose dimension is 4; if $F = (x_0^2 + x_1^2 + x_2^2 + x_3^2)^2$, then Q_2^{-2} is equal to the (direct) sum of

$$\begin{aligned}
&\{(0, x_3 u, -x_2 u, 0, 0, x_0 u) + \operatorname{im} \partial_{\boldsymbol{v}} \mid u \in S_1\},\\
&\{(x_3 v, 0, -x_1 v, 0, x_0 v, 0) + \operatorname{im} \partial_{\boldsymbol{v}} \mid v \in S_1\},\\
&\{(x_2 p, -x_1 p, 0, x_0 p, 0, 0) + \operatorname{im} \partial_{\boldsymbol{v}} \mid p \in S_1\},\\
&\{(0, 0, 0, x_3 q, -x_2 q, x_0 q) + \operatorname{im} \partial_{\boldsymbol{v}} \mid q \in S_1\},
\end{aligned}$$

and so $\dim Q_2^{-2} = 16$. We omit the computational details and list the dimensions of $H^2_{\mathrm{GS}}(\mathcal{A})_2$ of these examples in the right column of Table 1. It is obvious that the lower bound of $\dim H^2_{\mathrm{GS}}(\mathcal{A})_2$ is 1. However, in the general case, the authors do not know either the upper bound of $\dim H^2_{\mathrm{GS}}(\mathcal{A})_2$, or any gaps between the bound and 1.

Recall that when X is smooth, the Hodge numbers of X are defined to be $h^{p,q} = \dim H^q(X, \Omega_X^p)$. Let $\omega_X = \Omega_X^2$ be the canonical sheaf of X. Then $\omega_X \cong \mathcal{O}_X$ and by [4, Cor. 3.1.4],

$$H^2_{\mathrm{GS}}(\mathcal{A}) \cong HH^2(\omega_X) \cong H^2(X, \Omega_X^2) \oplus H^1(X, \Omega_X) \oplus H^0(X, \mathcal{O}_X).$$

The dimensions of the three summands are $h^{2,2} = 1$, $h^{1,1} = 20$, $h^{0,0} = 1$ respectively. So $\dim H^2_{\mathrm{GS}}(\mathcal{A})_r$ reaches its smallest possible values for $r = 0$, 1, 2 if X is smooth.

The converse is not true, as there indeed exist non-smooth surfaces with $\dim H^2_{\mathrm{GS}}(\mathcal{A})_1 = 20$ and $\dim H^2_{\mathrm{GS}}(\mathcal{A})_0 = \dim H^2_{\mathrm{GS}}(\mathcal{A})_2 = 1$. Let us give two examples here.

Example 6.1. *Let* $F = x_0^4 + x_1^4 + x_2^4 - 4x_2 x_3^3 + 3x_3^4$. *We know* $\boldsymbol{u} = (4x_0^3, 4x_1^3, 4(x_2^3 - x_3^3), -12x_3^2(x_2 - x_3))$. *A direct computation shows that* $\dim P_4^0 = 19$ *and* $\dim Q_2^{-2} = 0$. *Note that the surface has three isolated singularities* $(0 : 0 : 1 : \zeta^r)$ *for* $r = 0$, 1, 2 *where* ζ *is a primitive third root of* 1. *Furthermore, we have* $\dim H^1(X, \mathcal{T}_X) = 11$, *in accordance with Theorem 6.3.*

Example 6.2. *The Kummer surfaces* K_μ *are a family of quartic surfaces given by*

$$F = (x_0^2 + x_1^2 + x_2^2 - \mu^2 x_3^2)^2 - \lambda pqrs$$

where

$$\lambda = \frac{3\mu^2 - 1}{3 - \mu^2}$$

and p, q, r, s *are the tetrahedral coordinates*

$$p = x_3 - x_2 - \sqrt{2}x_0, \quad q = x_3 - x_2 + \sqrt{2}x_0,$$
$$r = x_3 + x_2 + \sqrt{2}x_1, \quad s = x_3 + x_2 - \sqrt{2}x_1.$$

When $\mu^2 \neq 1/3,\ 1,$ *or* 3, K_μ *has* 16 *isolated singularities which are ordinary double points. In this case, one can check that* $\boldsymbol{u}$ *is a regular sequence in* R. *Thus* $\dim P_4^0 = 19$ *and* $\dim Q_2^{-2} = 0$. *We also have* $\dim H^1(X, \mathcal{T}_X) = 1$, *in accordance with Theorem 6.3.*

The examples given above with $\dim H^0(X, \wedge^2\mathcal{T}_X) = \dim H^2_{\mathrm{GS}}(\mathcal{A})_2 = 1$ are all integral, and vice versa. We will give two examples to show this condition is neither necessary nor sufficient for integrality of X.

Example 6.3. *Let* $F = (x_0^2 + x_1^2 + 2x_2^2)(x_0^2 + x_1^2 + 2x_3^2)$. *We can easily prove* $Q_2^{-2} = 0$ *and hence* $\dim H^0(X, \wedge^2\mathcal{T}_X) = 1$. *However, this is not integral.*

Example 6.4. *Let* $F = x_0^4 + x_1^3 x_2$. *This gives rise to an integral scheme. But* Q_2^{-2} *is spanned by*

$$(0, 0, 0, 0, x_1 u, -x_2 u) + \operatorname{im}\partial_{\boldsymbol{v}}, \quad u \in \{x_0, x_1, x_2, x_3\}$$

which is 4*-dimensional.*

According to our general results, for a smooth K3 surface, we have $P_4^0 = E_{\mathrm{mult}} \subseteq H^2_{\mathrm{GS}}(\mathcal{A})_1 = E_{\mathrm{res}}$ and $\dim P_4^0 = 19$. To end this section, let us present the resulting two different deformation interpretations of Hochschild 2-classes in P_4^0 for the Fermat quartic surface, i.e. the first example in Table 1. Since $\boldsymbol{u} = (4x_0^3, 4x_1^3, 4x_2^3, 4x_3^3)$, P_4^0 has a basis

$$\{x_0^{i_0} x_1^{i_1} x_2^{i_2} x_3^{i_3} \mid i_0 + i_1 + i_2 + i_3 = 4,\ 0 \leq i_0, i_1, i_2, i_3 \leq 2\}.$$

We fix the generators and relations of $\mathcal{A}(V)$ for all $V \in \mathfrak{V}$ as follows:

$$A_1 = k[y_0, y_2, y_3]/(y_0^4 + y_2^4 + y_3^4 + 1), \qquad A_2 = k[y_0, y_1, y_3]/(y_0^4 + y_1^4 + y_3^4 + 1),$$
$$A_3 = k[y_0, y_1, y_2]/(y_0^4 + y_1^4 + y_2^4 + 1), \qquad A_{12} = k[y_0, y_1, y_3, y_1^{-1}]/(y_0^4 + y_1^4 + y_3^4 + 1),$$
$$A_{13} = k[y_0, y_1, y_2, y_1^{-1}]/(y_0^4 + y_1^4 + y_2^4 + 1), \qquad A_{23} = k[y_0, y_1, y_2, y_2^{-1}]/(y_0^4 + y_1^4 + y_2^4 + 1),$$
$$A_{123} = k[y_0, y_1, y_2, y_1^{-1}, y_2^{-1}]/(y_0^4 + y_1^4 + y_2^4 + 1).$$

For any basis element $x_0^{i_0} x_1^{i_1} x_2^{i_2} x_3^{i_3} \in P_4^0$, there is a deformation $(m, 0, 0)$ of $\mathcal{A}$ given by

$$m_{V_1} = y_0^{i_0} y_2^{i_2} y_3^{i_3} {}^{\circ}\mu, \quad m_{V_2} = y_0^{i_0} y_1^{i_1} y_3^{i_3} {}^{\circ}\mu, \quad m_{V_3} = y_0^{i_0} y_1^{i_1} y_2^{i_2} {}^{\circ}\mu,$$
$$m_{V_{12}} = y_0^{i_0} y_1^{i_1} y_3^{i_3} {}^{\circ}\mu, \quad m_{V_{13}} = y_0^{i_0} y_1^{i_1} y_2^{i_2} {}^{\circ}\mu, \quad m_{V_{23}} = y_0^{i_0} y_1^{i_1} y_2^{i_2} {}^{\circ}\mu,$$
$$m_{V_{123}} = y_0^{i_0} y_1^{i_1} y_2^{i_2} {}^{\circ}\mu.$$

We remark that although the same notation ${}^{\circ}\mu$ is used, it stands for Hochschild 2-cocycles of individual algebras.

Since in A_1 one has

$$1 = 4y_0^3\left(-\frac{1}{4}y_0\right) + 4y_2^3\left(-\frac{1}{4}y_2\right) + 4y_3^3\left(-\frac{1}{4}y_3\right),$$

it follows that

$${}^{\circ}\mu = d_{\mathrm{Hoch}}\left(-\frac{1}{4}y_0\frac{{}^{\circ}\partial}{{}^{\circ}\partial y_0} - \frac{1}{4}y_2\frac{{}^{\circ}\partial}{{}^{\circ}\partial y_2} - \frac{1}{4}y_3\frac{{}^{\circ}\partial}{{}^{\circ}\partial y_3}\right).$$

Similarly, for A_2 and A_3, we respectively have

$${}^{\circ}\mu = d_{\mathrm{Hoch}}\left(-\frac{1}{4}y_0\frac{{}^{\circ}\partial}{{}^{\circ}\partial y_0} - \frac{1}{4}y_1\frac{{}^{\circ}\partial}{{}^{\circ}\partial y_1} - \frac{1}{4}y_3\frac{{}^{\circ}\partial}{{}^{\circ}\partial y_3}\right),$$

$$^{\circ}\mu = d_{\mathrm{Hoch}}\Big(-\frac{1}{4}y_0\frac{^{\circ}\partial}{^{\circ}\partial y_0} - \frac{1}{4}y_1\frac{^{\circ}\partial}{^{\circ}\partial y_1} - \frac{1}{4}y_2\frac{^{\circ}\partial}{^{\circ}\partial y_2}\Big).$$

The three preimages are denoted by s_1, s_2, s_3. By abuse of notation, they also denote 1-cochains of the algebras A_{12}, A_{13} and so on. Then we have

$$\begin{aligned} m_{V_1} &= d_{\mathrm{Hoch}}(y_0^{i_0}y_2^{i_2}y_3^{i_3}s_1), & m_{V_2} &= d_{\mathrm{Hoch}}(y_0^{i_0}y_1^{i_1}y_3^{i_3}s_2),\\ m_{V_3} &= d_{\mathrm{Hoch}}(y_0^{i_0}y_1^{i_1}y_2^{i_2}s_3), & m_{V_{12}} &= d_{\mathrm{Hoch}}(y_0^{i_0}y_1^{i_1}y_3^{i_3}s_2),\\ m_{V_{13}} &= d_{\mathrm{Hoch}}(y_0^{i_0}y_1^{i_1}y_2^{i_2}s_3), & m_{V_{23}} &= d_{\mathrm{Hoch}}(y_0^{i_0}y_1^{i_1}y_2^{i_2}s_3),\\ m_{V_{123}} &= d_{\mathrm{Hoch}}(y_0^{i_0}y_1^{i_1}y_2^{i_2}s_3). \end{aligned}$$

We choose a map $\lambda\colon \mathcal{V}\to\mathcal{U}$ by $\lambda(V_{j_1\cdots j_r}) = U_{j_r}$ if $j_1<\cdots<j_r$. We thus obtain an equivalent deformation $(0,f,0)$ whose nonzero components of f are

$$\begin{aligned} f_{V_{12}\subseteq V_1} &= y_0^{i_0}y_1^{i_1}y_3^{i_3}s_2\circ\rho^{V_1}_{V_{12}} - \rho^{V_1}_{V_{12}}\circ y_0^{i_0}y_2^{i_2}y_3^{i_3}s_1,\\ f_{V_{13}\subseteq V_1} &= y_0^{i_0}y_1^{i_1}y_2^{i_2}s_3\circ\rho^{V_1}_{V_{13}} - \rho^{V_1}_{V_{13}}\circ y_0^{i_0}y_2^{i_2}y_3^{i_3}s_1,\\ f_{V_{23}\subseteq V_2} &= y_0^{i_0}y_1^{i_1}y_2^{i_2}s_3\circ\rho^{V_2}_{V_{23}} - \rho^{V_2}_{V_{23}}\circ y_0^{i_0}y_1^{i_1}y_3^{i_3}s_2,\\ f_{V_{123}\subseteq V_1} &= y_0^{i_0}y_1^{i_1}y_2^{i_2}s_3\circ\rho^{V_1}_{V_{123}} - \rho^{V_1}_{V_{123}}\circ y_0^{i_0}y_2^{i_2}y_3^{i_3}s_1,\\ f_{V_{123}\subseteq V_2} &= y_0^{i_0}y_1^{i_1}y_2^{i_2}s_3\circ\rho^{V_2}_{V_{123}} - \rho^{V_2}_{V_{123}}\circ y_0^{i_0}y_1^{i_1}y_3^{i_3}s_2,\\ f_{V_{123}\subseteq V_{12}} &= y_0^{i_0}y_1^{i_1}y_2^{i_2}s_3\circ\rho^{V_{12}}_{V_{123}} - \rho^{V_{12}}_{V_{123}}\circ y_0^{i_0}y_1^{i_1}y_3^{i_3}s_2. \end{aligned}$$

Appendix A. Converse of Hochschild-Kostant-Rosenberg theorem

In this appendix, we give a proof of Theorem 6.3 for complete intersections X instead of hypersurfaces. This proof is adapted from the referee's report.

Let X be a closed subscheme of a nonsingular variety Y over k. Recall that X is a local complete intersection in Y if the ideal sheaf $\mathcal{I}$ of X in Y can be generated by $\operatorname{codim}(X,Y)$ elements at every point. As we discussed in §4.2, the cotangent complex $\mathbb{L}_{X/k}$ is concentrated in degrees 0 and -1 with

$$\mathbb{L}^0_{X/k} = \imath^*\Omega_Y,\quad \mathbb{L}^{-1}_{X/k} = \mathcal{I}/\mathcal{I}^2,$$

where $\imath$ is the closed immersion $X\hookrightarrow Y$. By definition, $\mathbb{L}_{X/k}$ is a complex of locally free sheaves of finite rank. As the same argument at the end of §4, we have

$$\operatorname{Ext}^p_X(\wedge^q\mathbb{L}_{X/k},\mathcal{O}_X)\cong\mathbb{H}^{p+q}(\wedge^q\mathbb{L}^\vee_{X/k}).$$

So Buchweitz-Flenner's formula for $HH^2(X)$ becomes

$$HH^2(X) = \mathbb{H}^2(\mathcal{O}_X)\oplus\mathbb{H}^1(\mathbb{L}^\vee_{X/k})\oplus\mathbb{H}^0(\wedge^2\mathbb{L}^\vee_{X/k}).$$

Since $(\mathcal{I}/\mathcal{I}^2)^\vee = \mathcal{N}_{X/Y}$ is the normal sheaf, $\mathbb{L}^\vee_X$ is the two-term complex

$$\mathcal{T}_Y|_X \xrightarrow{\partial^\top} \mathcal{N}_{X/Y}$$

with cohomology sheaves $\mathcal{H}^0(\mathbb{L}^\vee_X) = \mathcal{T}_X$ and $\mathcal{H}^1(\mathbb{L}^\vee_X) =: \mathcal{C}$.

Theorem A.1. *Let X be a local complete intersection, and let all notations be as above. Assume the normal sheaf $\mathcal{N}_{X/Y}$ is globally generated. The following are equivalent:*

(1) *X is smooth.*
(2) *The HKR decomposition holds for all cohomology groups, i.e.*

$$HH^i(X)\cong\bigoplus_{p+q=i}H^p(X,\wedge^q\mathcal{T}_X),\quad\forall\, i\in\mathbb{N}.$$

(3) *The HKR decomposition holds for the second cohomology, i.e.*

$$HH^2(X) \cong H^0(X, \wedge^2 \mathcal{T}_X) \oplus H^1(X, \mathcal{T}_X) \oplus H^2(X, \mathcal{O}_X).$$

Proof. We only prove (3) ⇒ (1). Since X is smooth if and only if $\mathcal{C} = 0$, it suffices to prove $\mathcal{C} = 0$ from (3).

We have $\mathbb{H}^2(\mathcal{O}_X) \cong H^2(X, \mathcal{O}_X)$ and $\mathbb{H}^0(\wedge^2 \mathbb{L}^\vee_{X/k}) \cong H^0(X, \wedge^2 \mathcal{T}_X)$. Hence the middle direct summand $\mathbb{H}^1(\mathbb{L}^\vee_{X/k})$ is isomorphic to $H^1(X, \mathcal{T}_X)$. Apply $\mathrm{R}\Gamma$ to the exact triangle

$$\mathcal{T}_X \longrightarrow \mathbb{L}^\vee_X \longrightarrow \mathcal{C}[-1] \tag{A.1}$$

and then we get a long exact sequence

$$\begin{aligned} 0 &\longrightarrow H^0(X, \mathcal{T}_X) \longrightarrow \mathbb{H}^0(\mathbb{L}^\vee_X) \longrightarrow 0 \\ &\longrightarrow H^1(X, \mathcal{T}_X) \xrightarrow{\cong} \mathbb{H}^1(\mathbb{L}^\vee_X) \xrightarrow{\omega} H^0(X, \mathcal{C}) \\ &\longrightarrow H^2(X, \mathcal{T}_X) \longrightarrow \mathbb{H}^2(\mathbb{L}^\vee_X) \longrightarrow H^1(X, \mathcal{C}) \longrightarrow \cdots . \end{aligned}$$

By (3), we have $\omega = 0$.

Next, we claim that the natural map

$$\omega' \colon H^0(X, \mathcal{N}_{X/Y}) \longrightarrow H^0(X, \mathcal{C})$$

is zero. In fact, observe the commutative squares

$$\begin{array}{ccc} 0 & \longrightarrow & \mathcal{N}_{X/Y} \\ \downarrow & & \downarrow \mathrm{id} \\ \mathcal{T}_Y|_X & \xrightarrow{\partial^\top} & \mathcal{N}_{X/Y} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{C} \end{array}$$

where the lower square is nothing but the map of complexes $\mathbb{L}^\vee_X \to \mathcal{C}[-1]$ in the triangle (A.1). Taking $\mathbb{H}^1$ we see that the composition

$$H^0(X, \mathcal{N}_{X/Y}) \longrightarrow \mathbb{H}^1(\mathbb{L}^\vee_X) \xrightarrow{\omega} H^0(X, \mathcal{C})$$

is ω'. Thus if ω is zero then so is ω'.

Finally, let us prove $\mathcal{C} = 0$. Consider the commutative square of evaluation maps

$$\begin{array}{ccc} H^0(X, \mathcal{N}_{X/Y}) \otimes \mathcal{O}_X & \xrightarrow{\tau_1} & \mathcal{N}_{X/Y} \\ {\scriptstyle \omega' \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \tau_2} \\ H^0(X, \mathcal{C}) \otimes \mathcal{O}_X & \longrightarrow & \mathcal{C} \end{array}$$

The map τ_1 is surjective because $\mathcal{N}_{X/Y}$ is globally generated, and τ_2 is surjective by definition of $\mathcal{C}$. Thus $\mathcal{C} = 0$ follows from $\omega' = 0$, and the proof is finished. □

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